Zarankiewicz for axis-parallel boxes

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- What about bipartite H? ESS only tells $\mathrm{ex}(n, H) = o(n^2)$!

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Theorem (Bukh, 2024)

$$z(n, n; s, t) = \Omega_t\left(n^{2-\frac{1}{t}}\right) \text{ if } s > 10^t$$

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Theorem (Chen–Liu–Ye, 2025+)

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Geometric incidences

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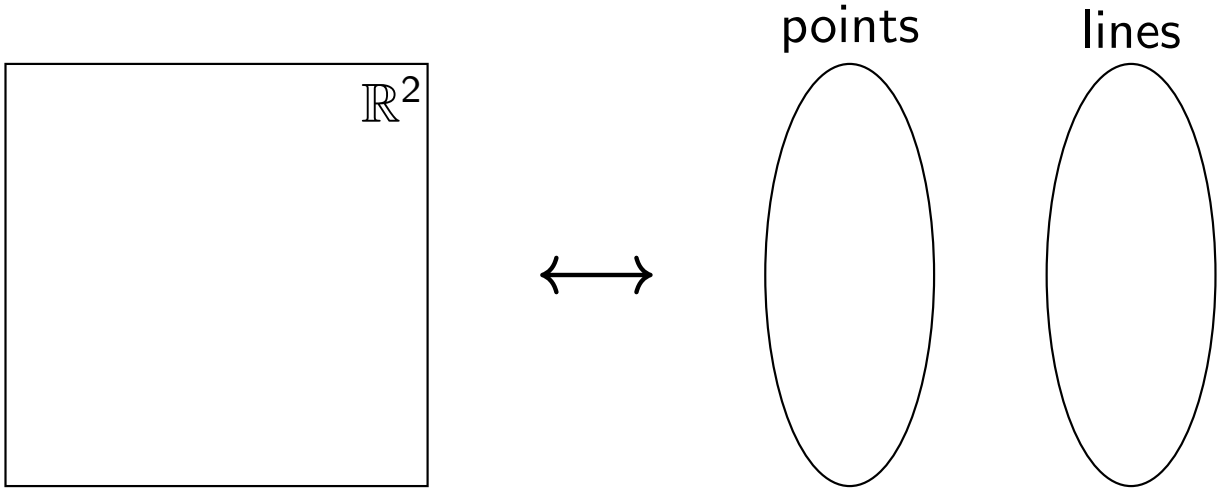

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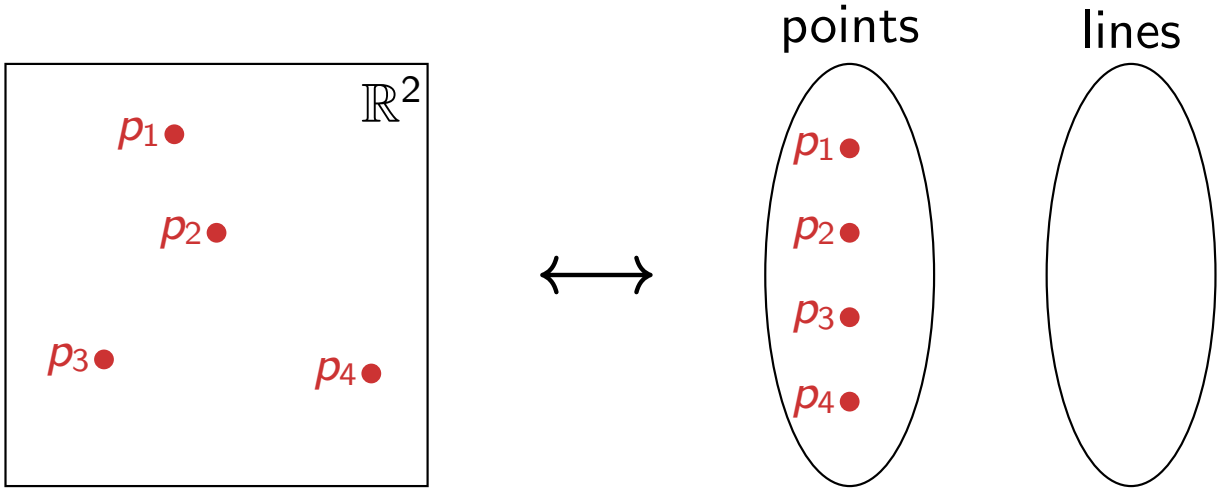

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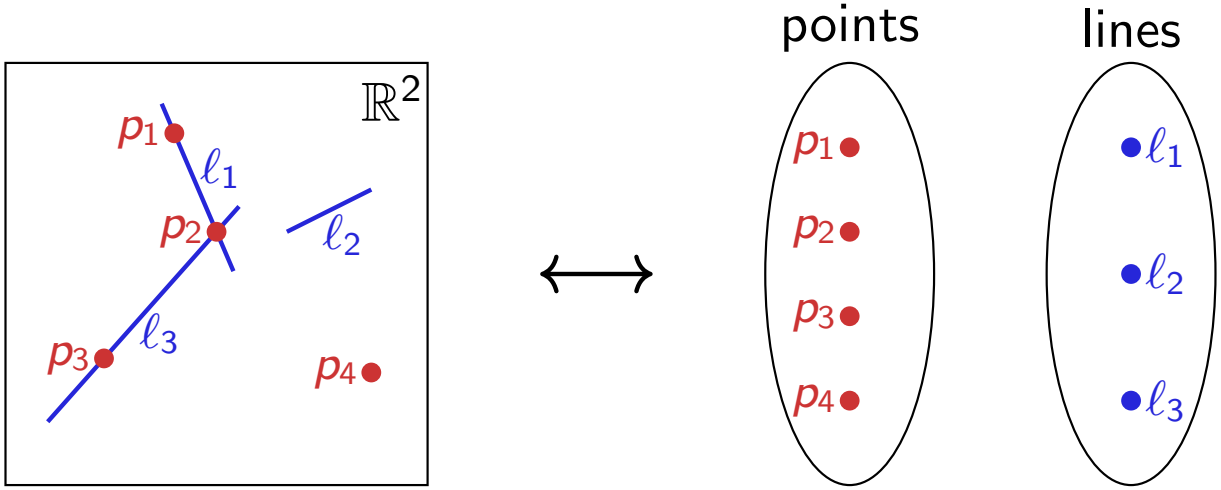

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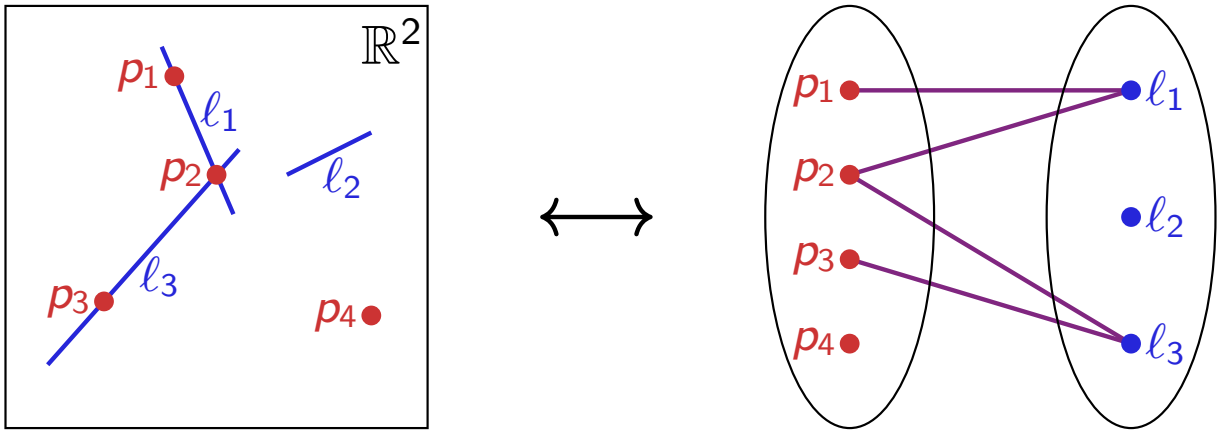

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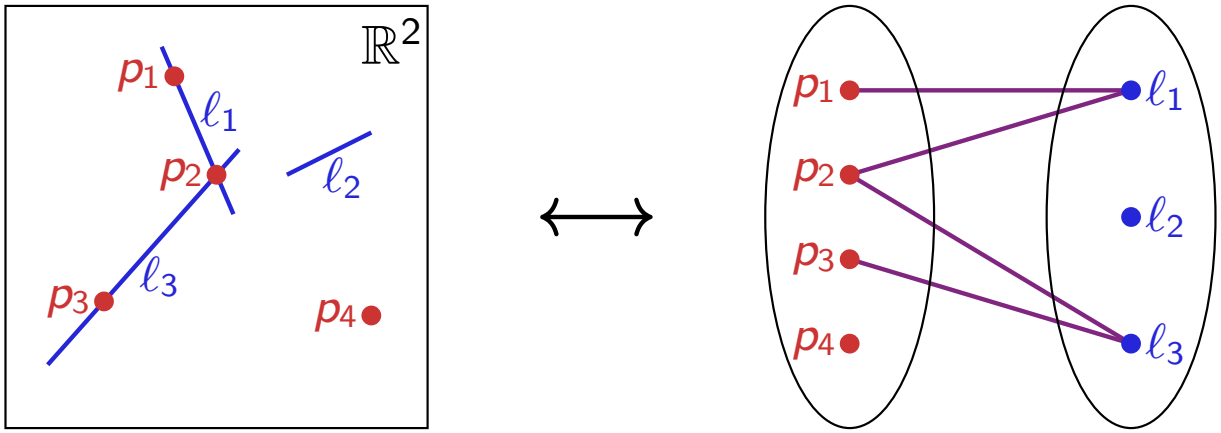

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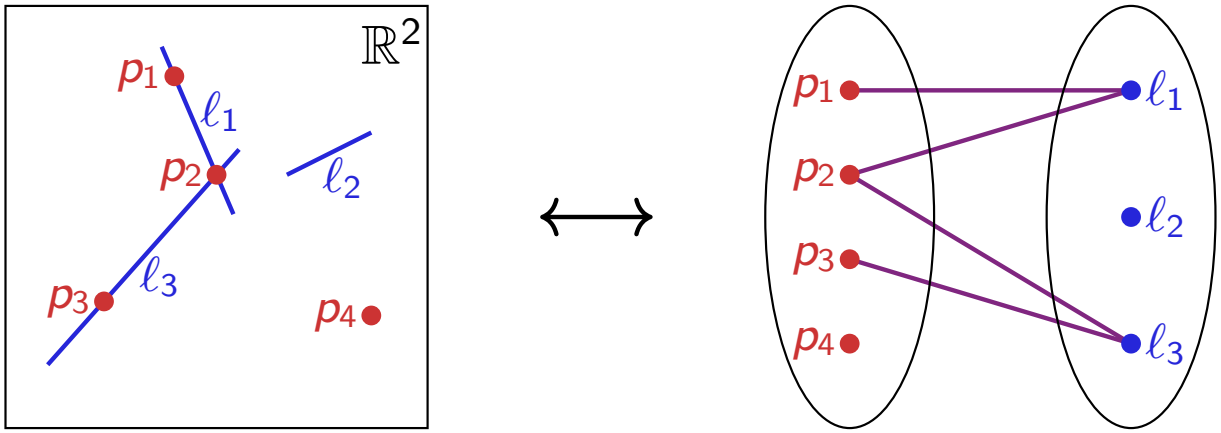

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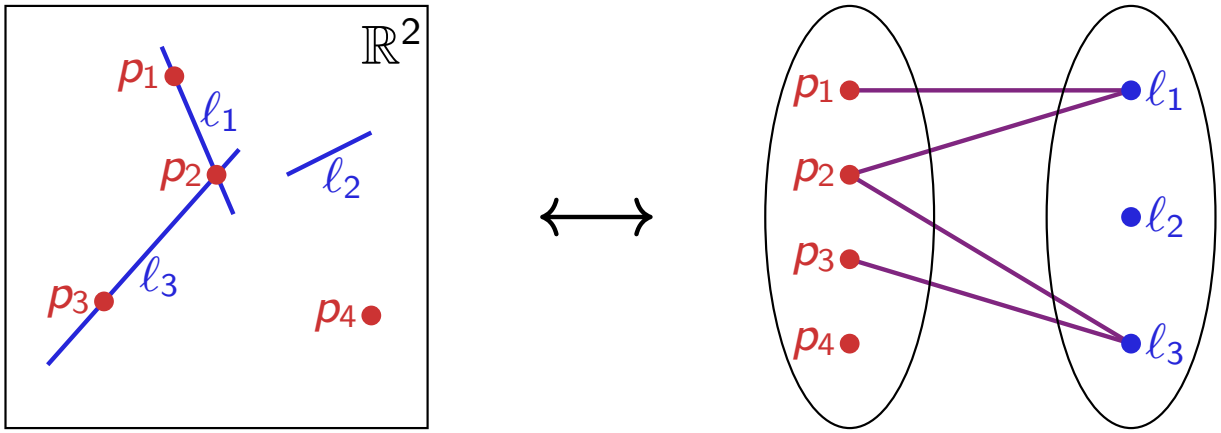

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- This $O(n^{4/3})$ bound is tight!

Axis-parallel boxes

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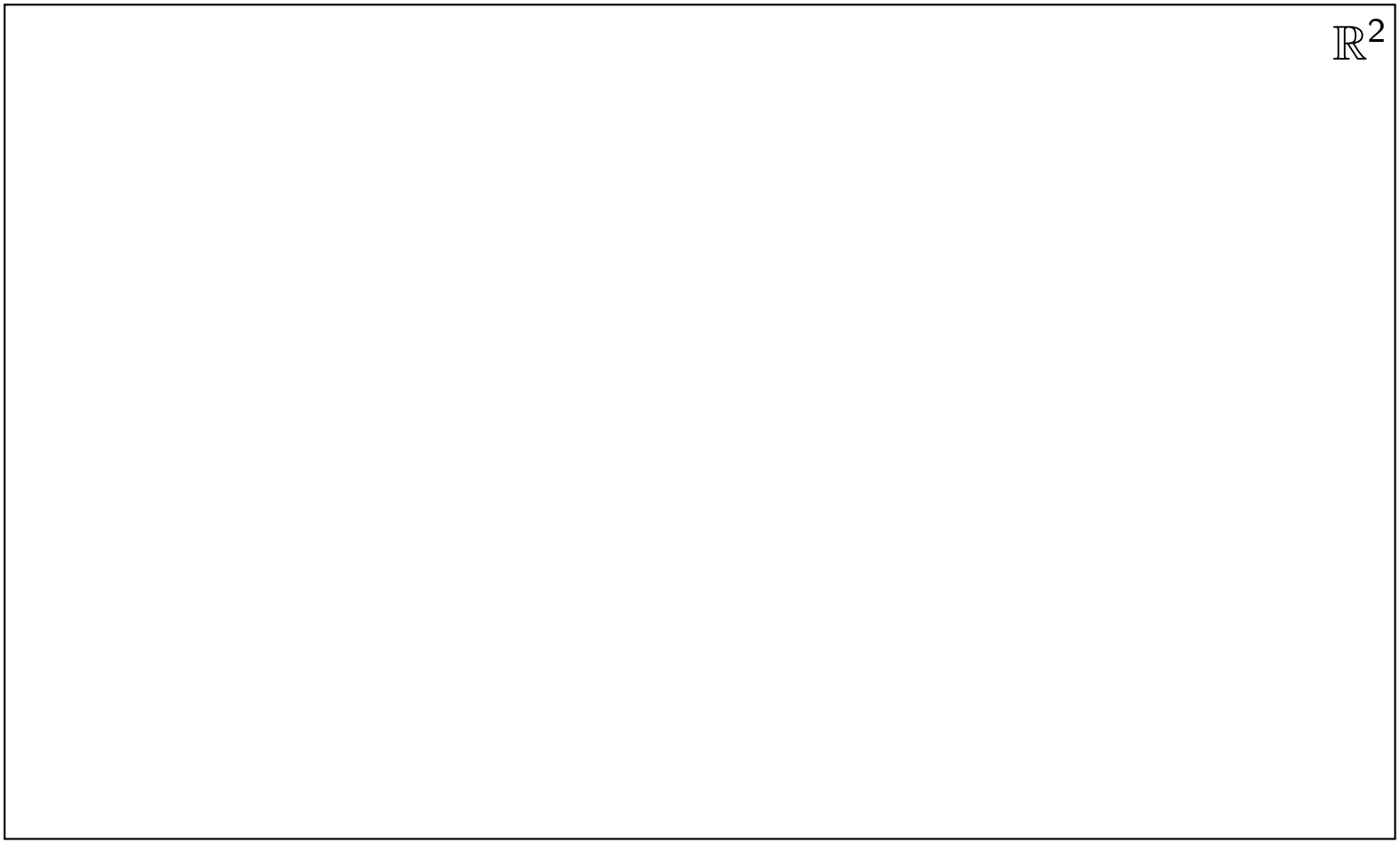

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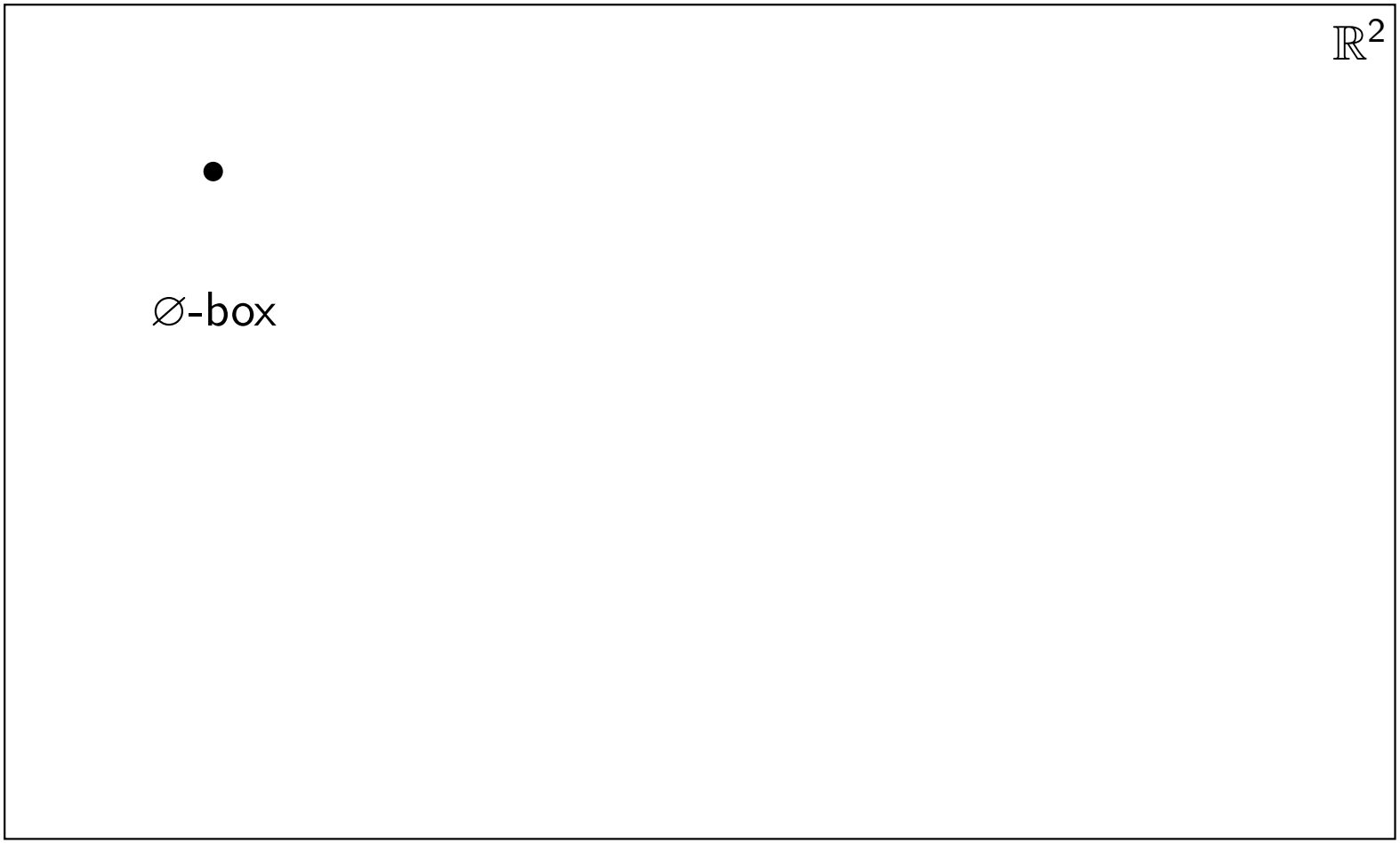

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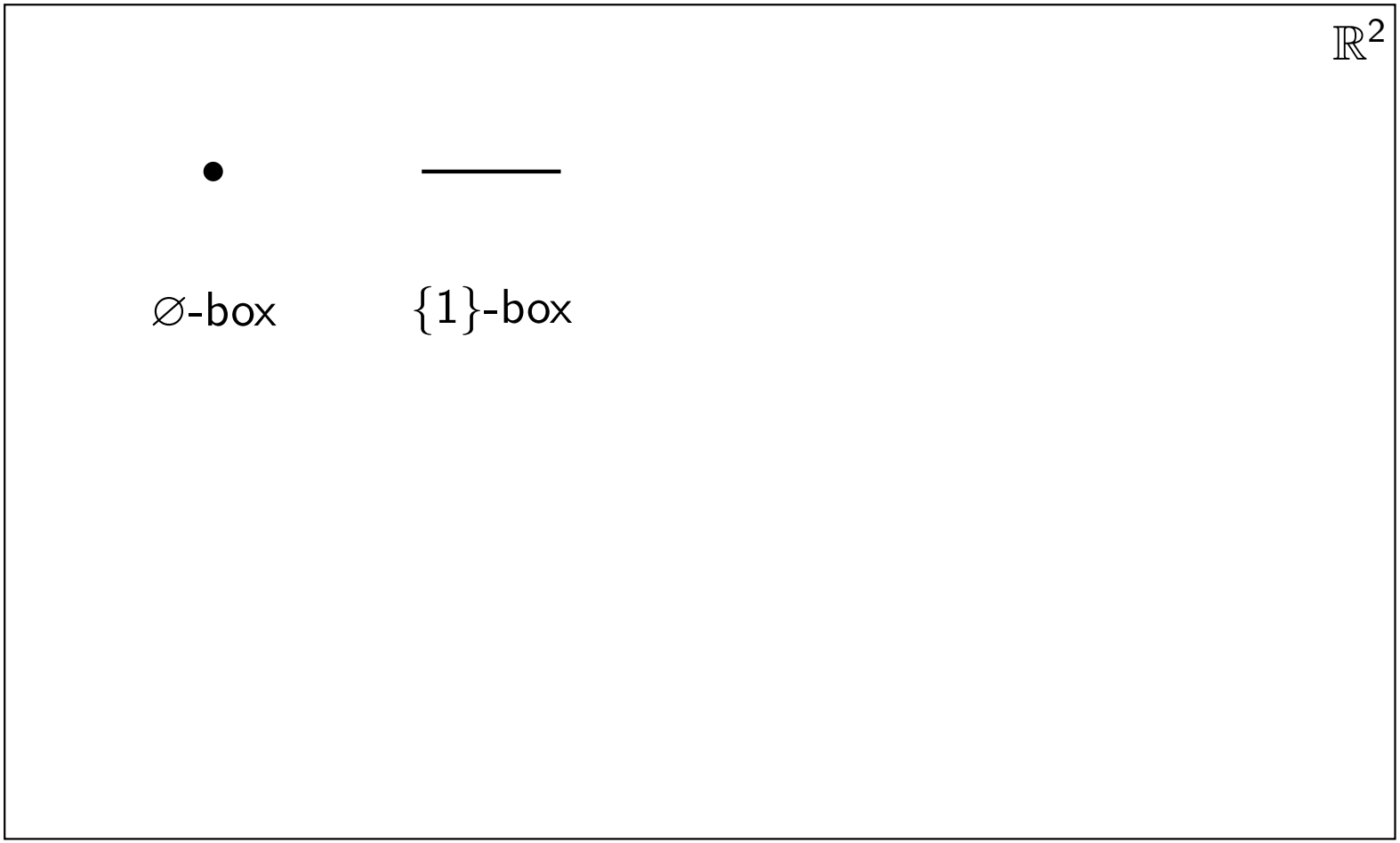

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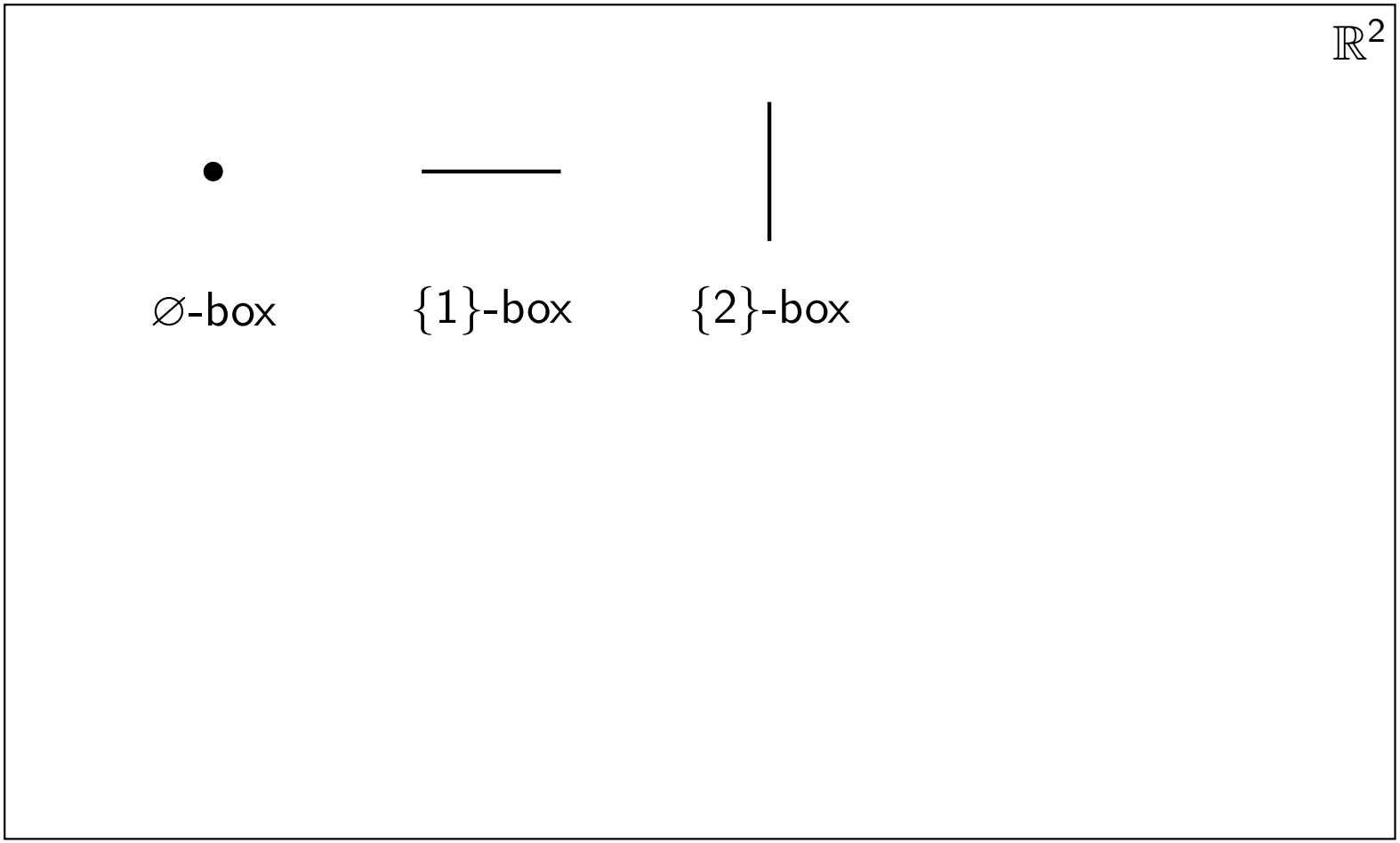

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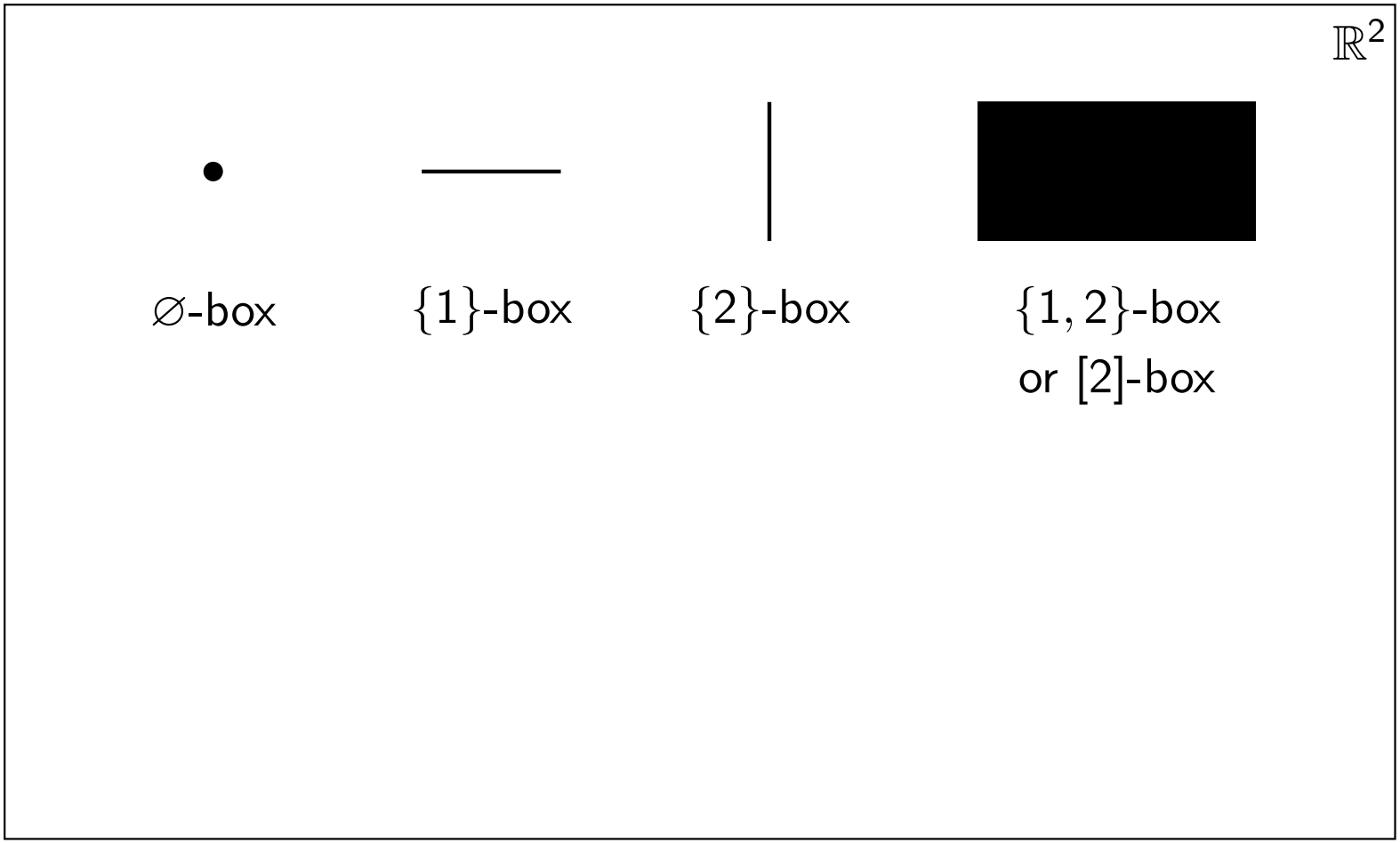

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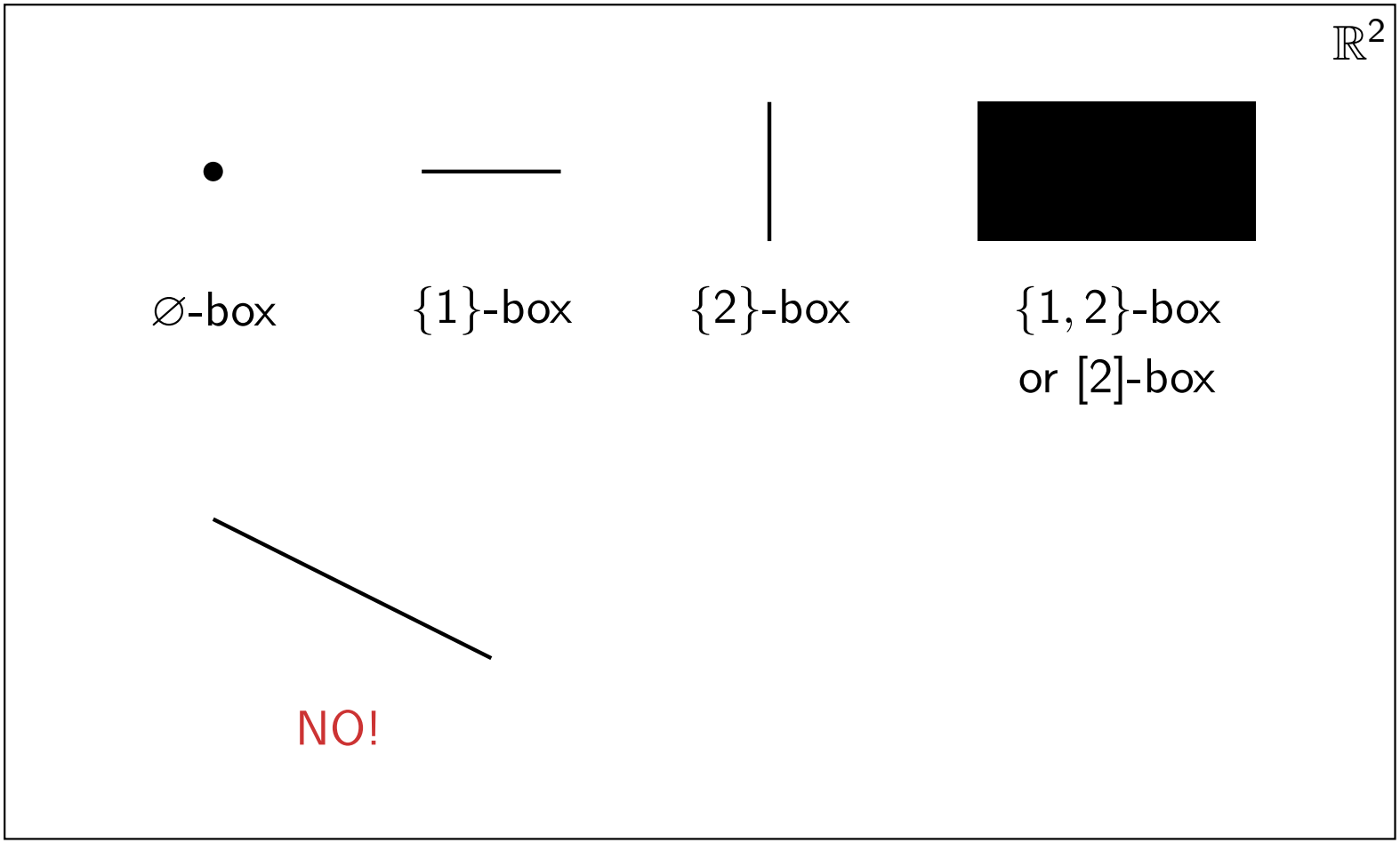

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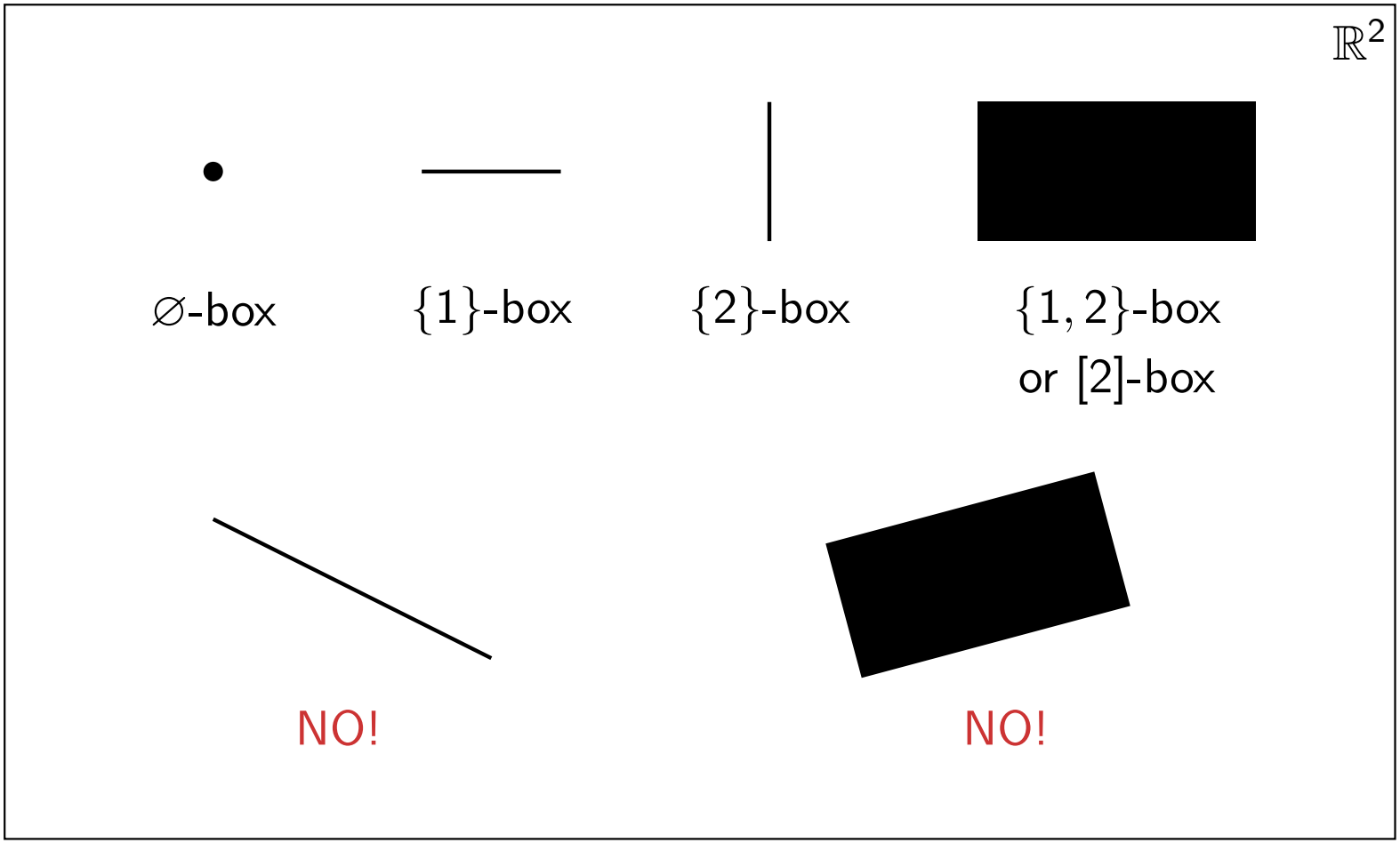

Box Zarankiewicz: Assume that $\vec{F} = (F_1, \ldots, F_r) \in \left(2^{[d]}\right)^r$ is an r-direction-vector. Let $\mathcal{B}_i$ be an F_i-family of boxes in $\mathbb{R}^d$ with $|\mathcal{B}_i| = n$. Forbidding $K^{(r)}_{t,\ldots,t}$, what is max # edges in their incidence hypergraph $\mathcal{H}$?

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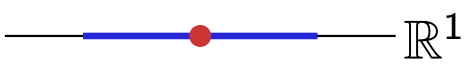

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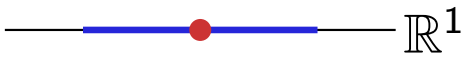

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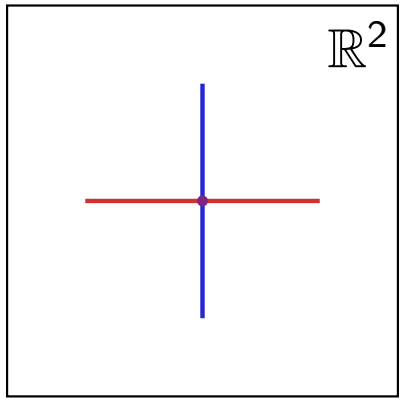

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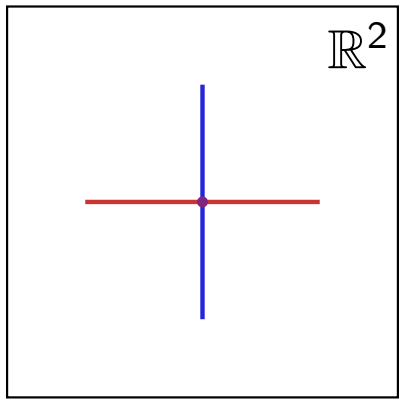

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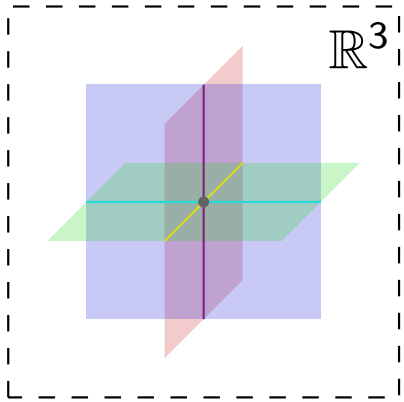

$\mathcal{I}_3(\{2,3\}, \{3,1\}, \{1,2\})$

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- **Zarankiewicz** $z^r(n;t) \approx n^{r-\varepsilon}$ yet **Box Z.** $z_d^{(F_1,\ldots,F_r)}(n;t) \approx n^{r-1}$

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- When does the extra logarithm factor happen?

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$$z_d^{\vec{F}}(n; t) = \begin{cases} \Theta_r(tn^{r-1}) & \text{if } \vec{F} \text{ is non-2-coherent,} \\ \Omega\left(tn^{r-1} \frac{\log n}{\log \log n}\right) & \text{if } \vec{F} \text{ is 2-coherent.} \end{cases}$$

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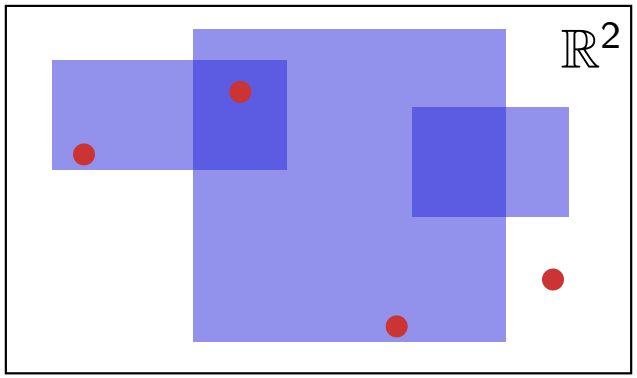

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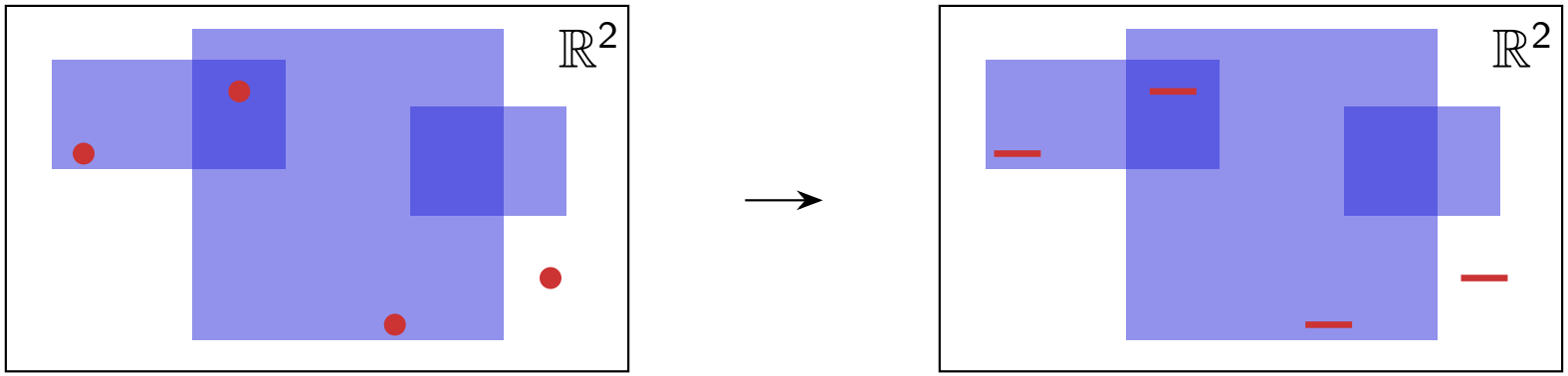

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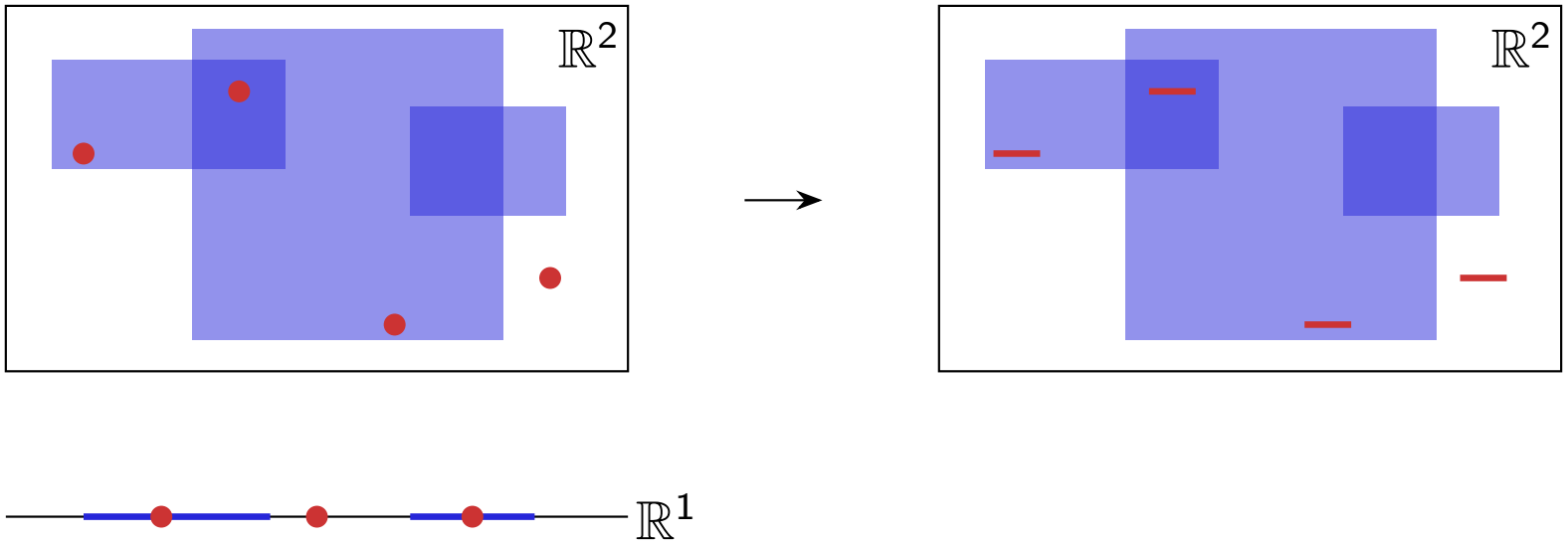

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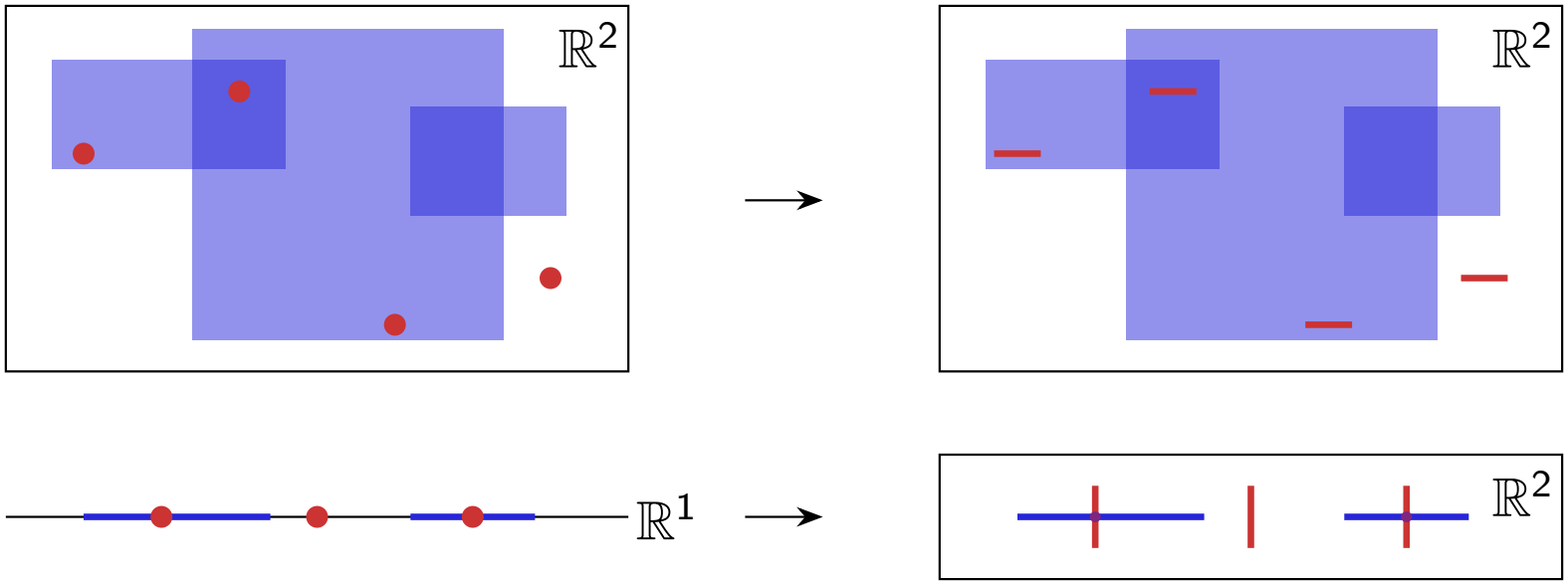

Upper bounds

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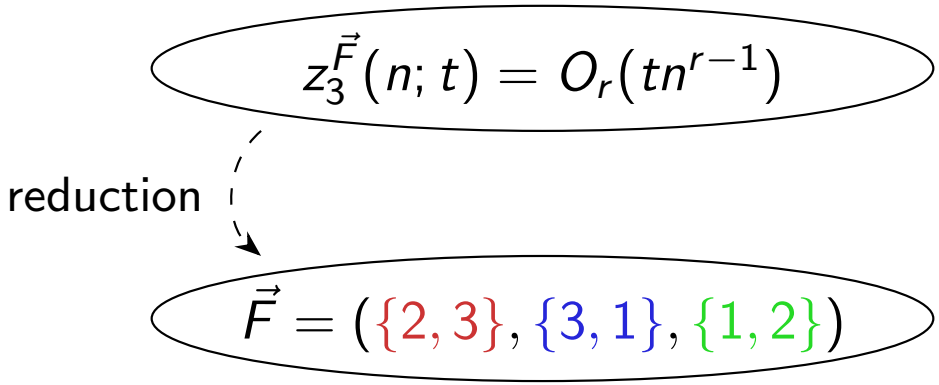
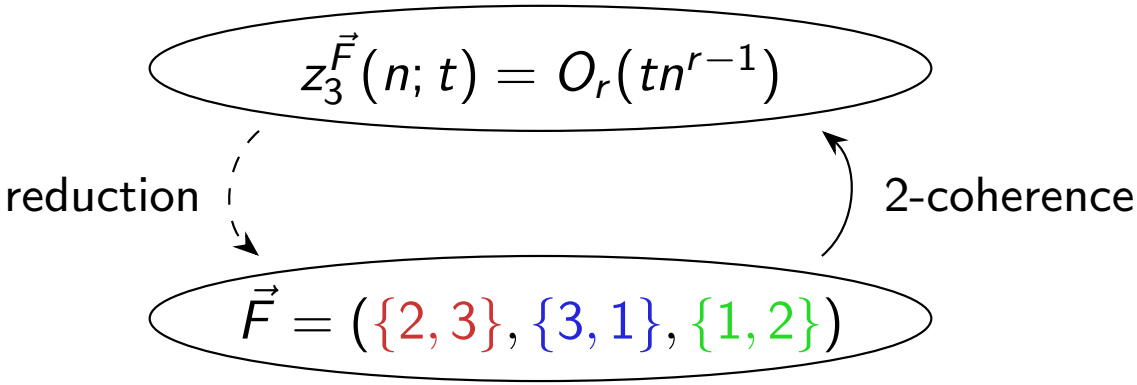
$$z_3^{\vec{F}}(n;t) = O_r(tn^{r-1})$$

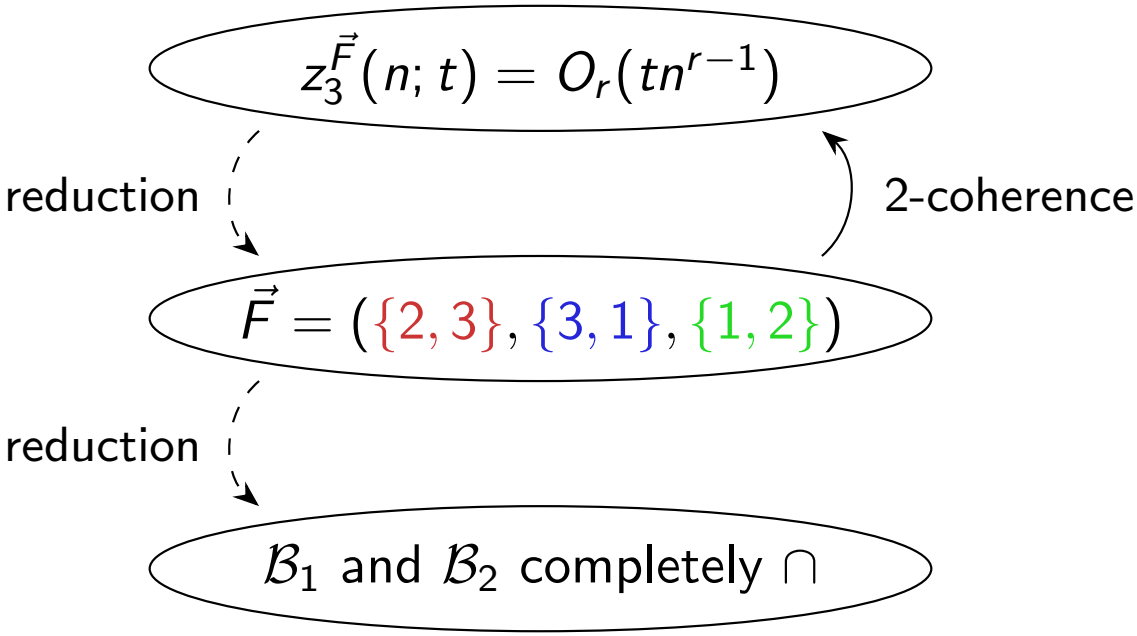
Upper bounds



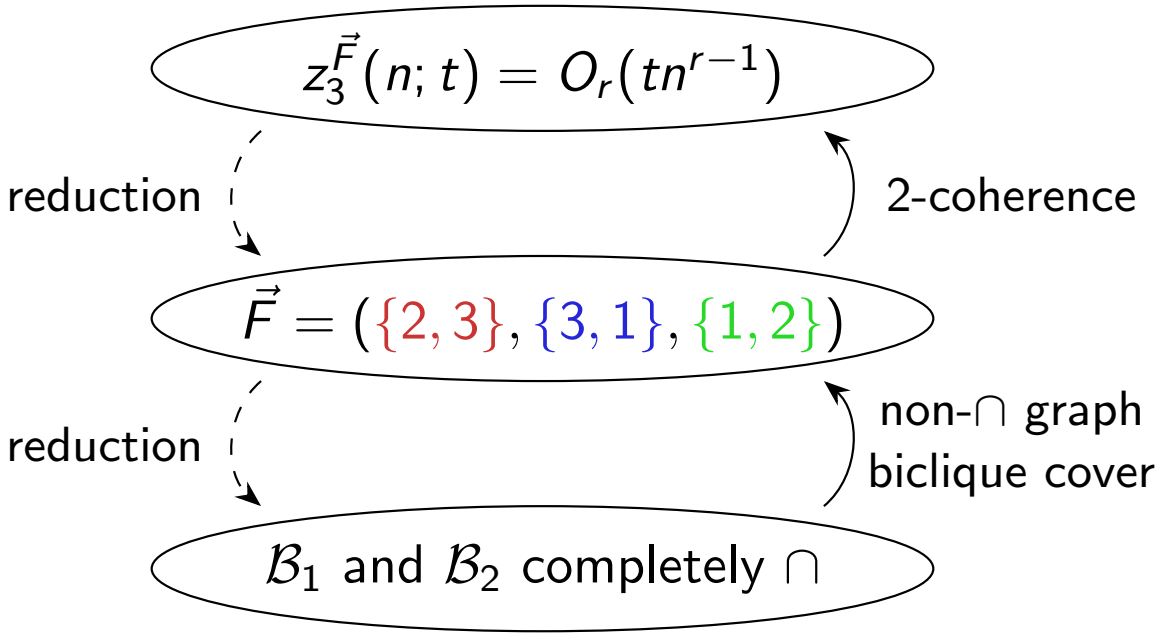
Upper bounds



Upper bounds



Upper bounds



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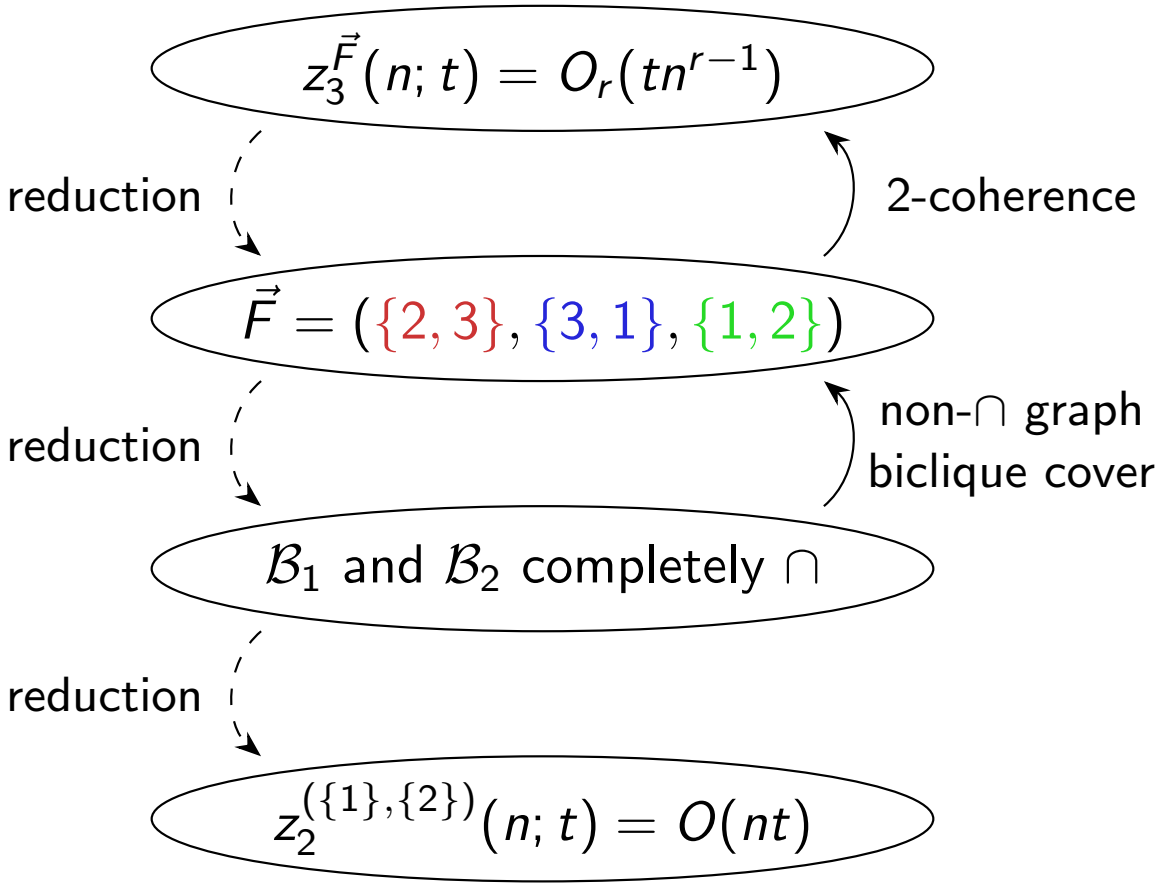

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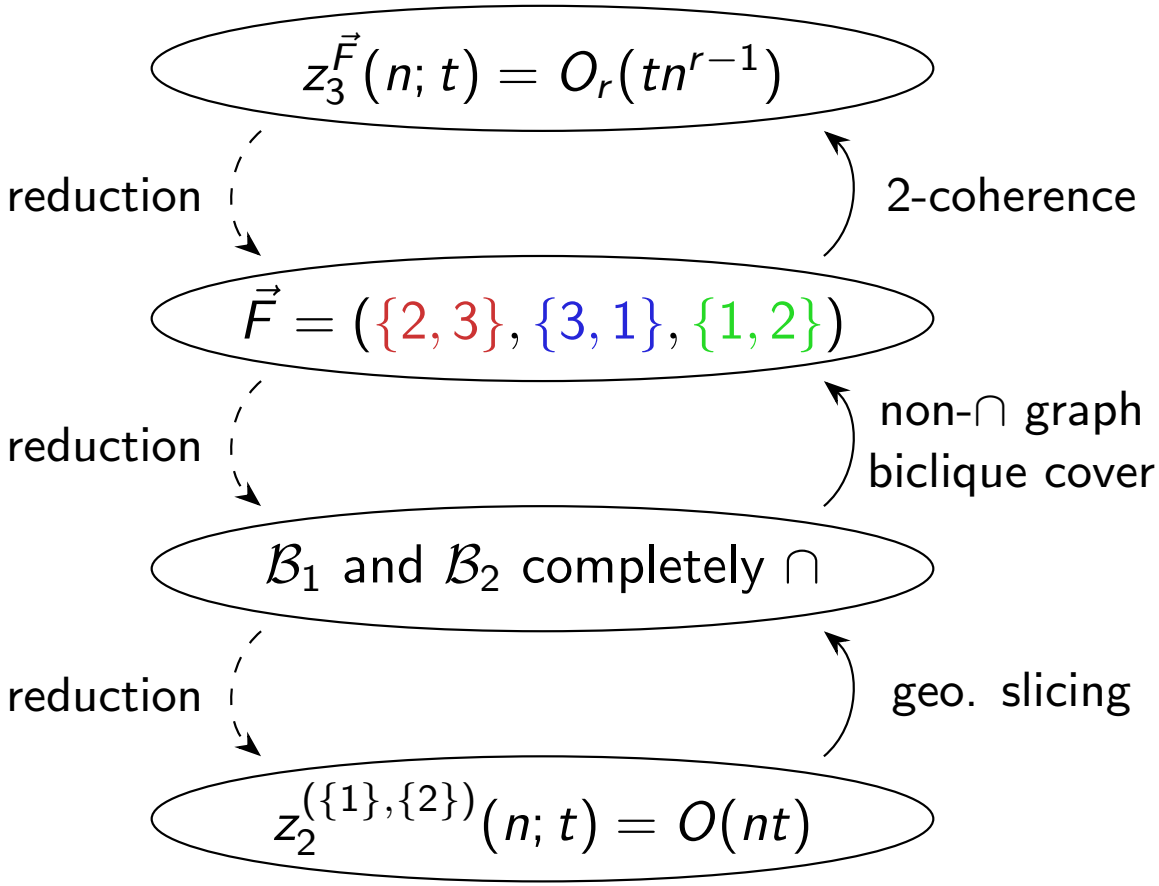

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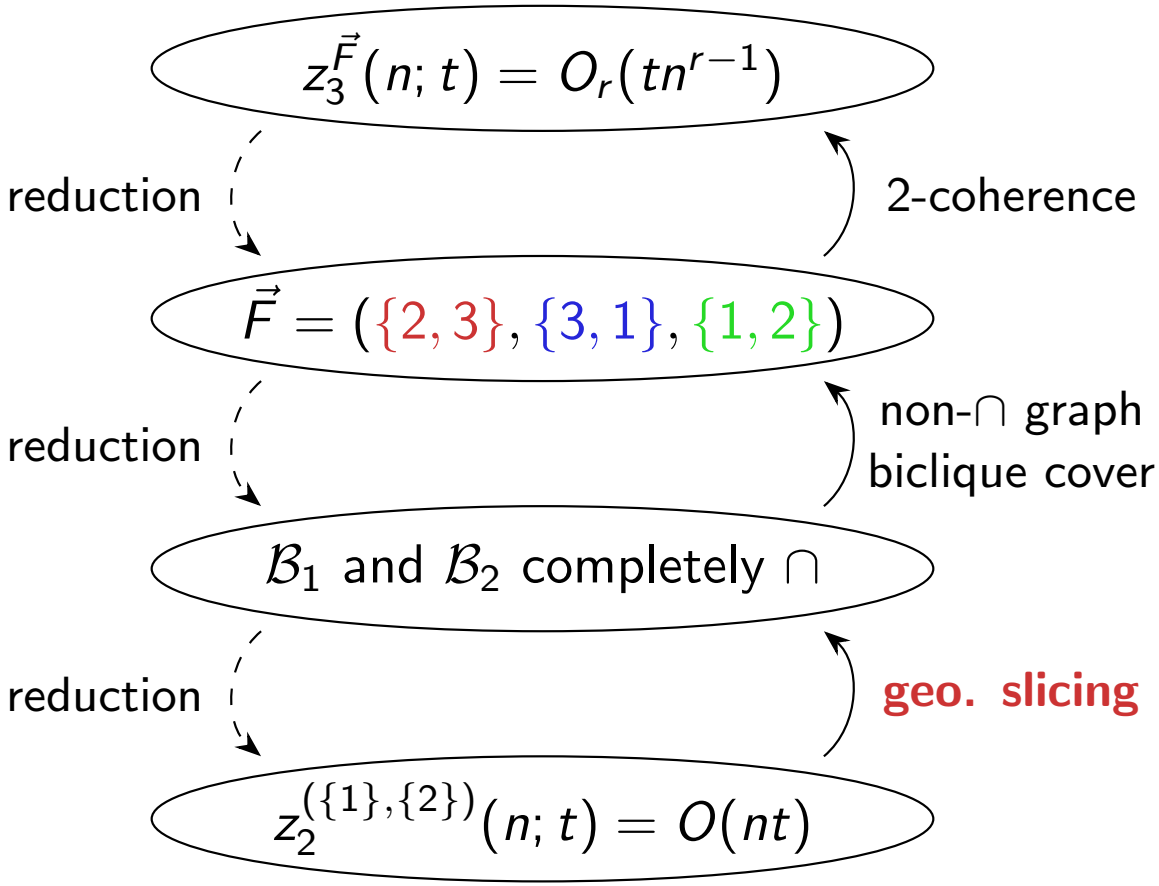
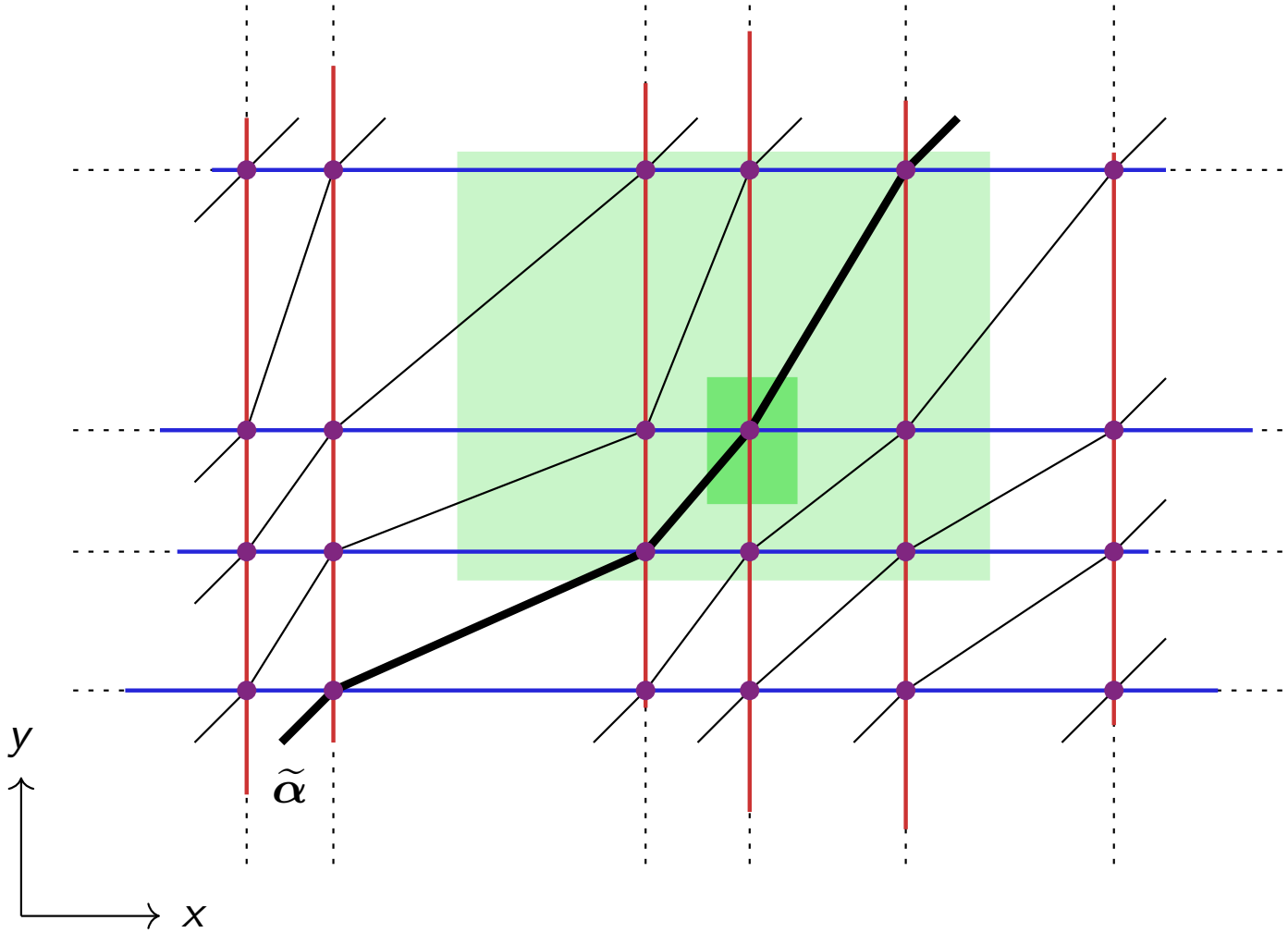
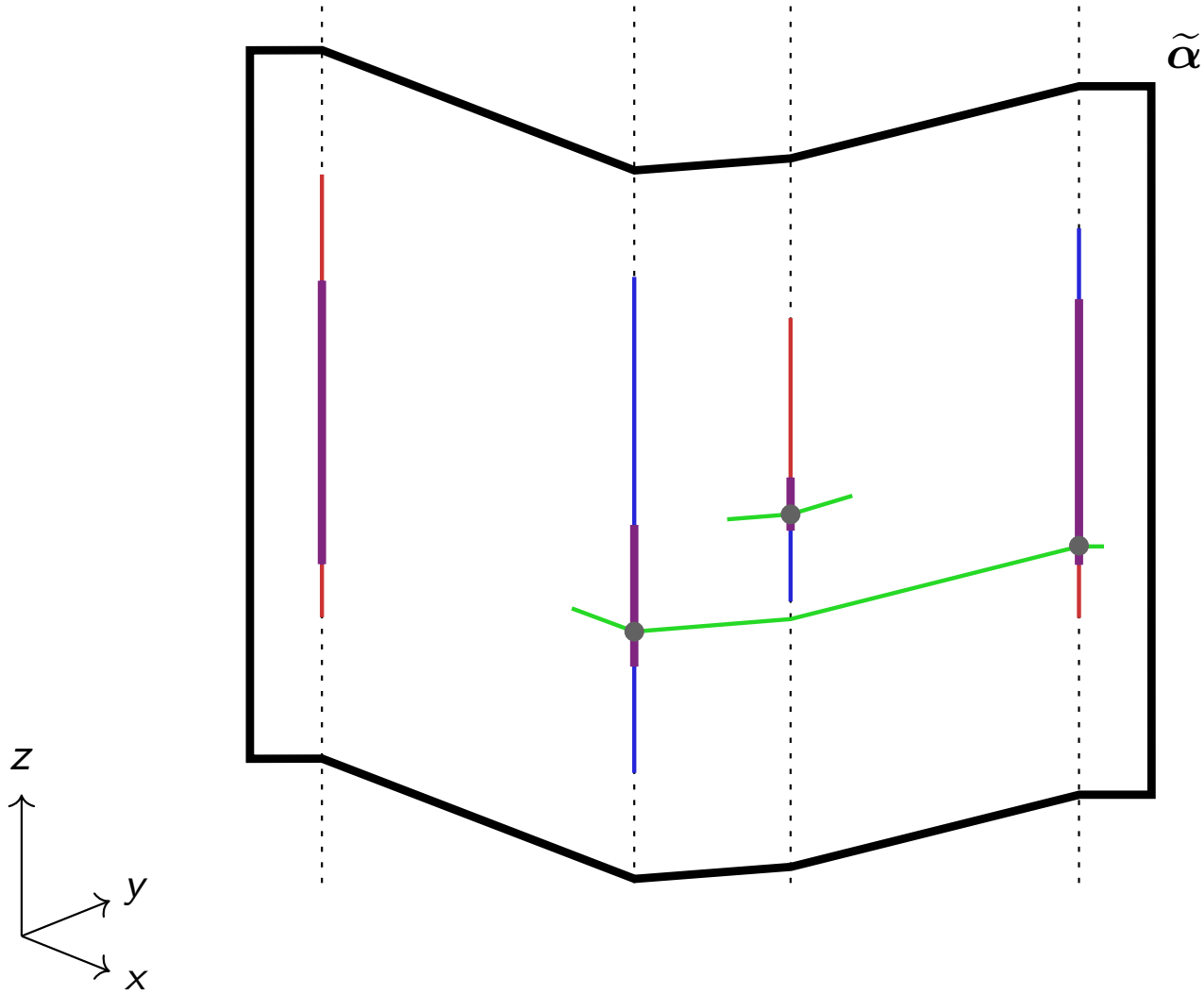


Geometric slicing



Geometric slicing



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