Rainbow even cycles

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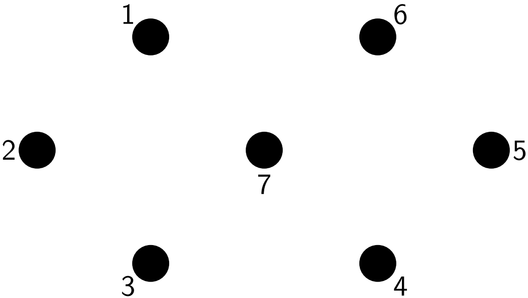
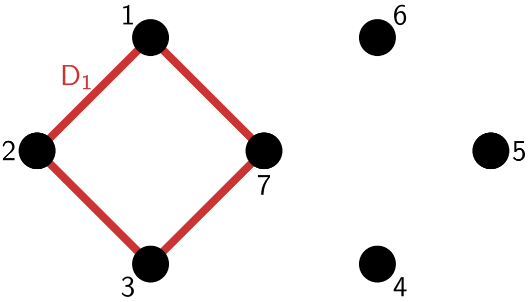
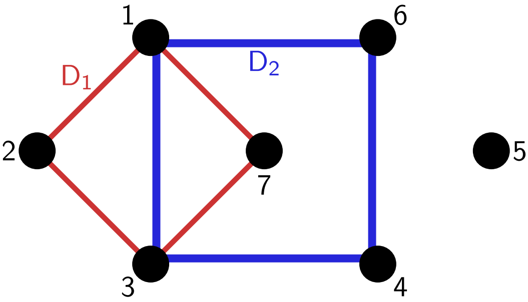


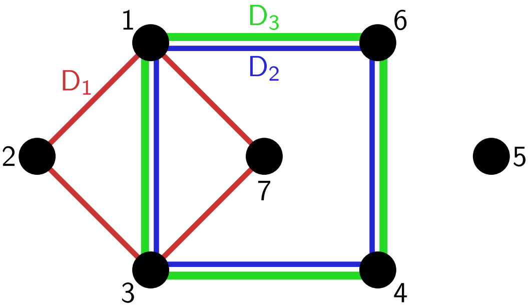
vertex set [7]



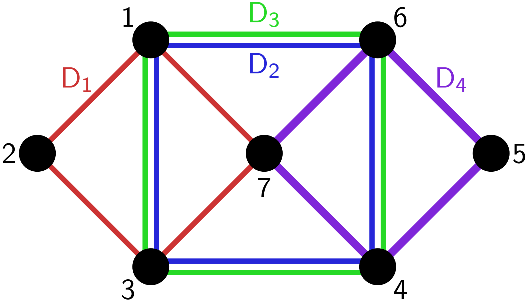
4-cycle D_1 on $1, 2, 3, 7$



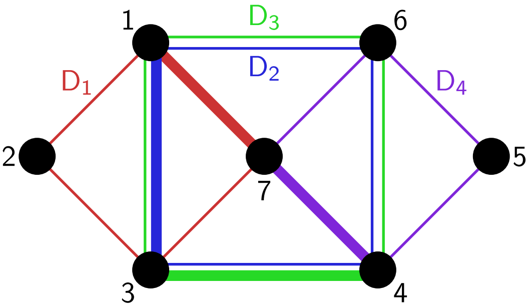
4-cycle D_2 on $1, 3, 4, 6$



4-cycle D_3 on $1, 3, 4, 6$



4-cycle D_4 on $4, 5, 6, 7$



rainbow 4-cycle on $1, 3, 4, 7$

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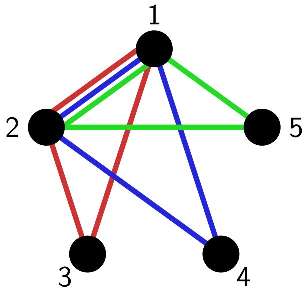

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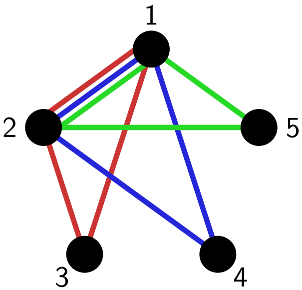

- three cycles on [5]
- no rainbow cycle

Theorem (Aharoni–Briggs–Holzman–Jiang, 2021)

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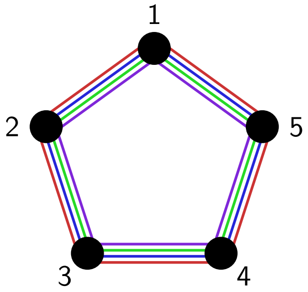

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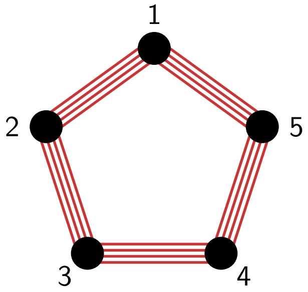

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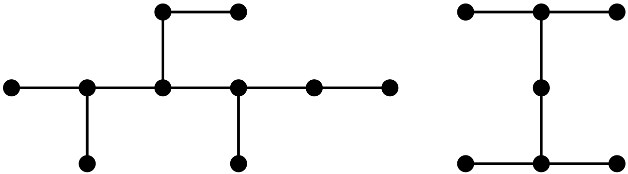

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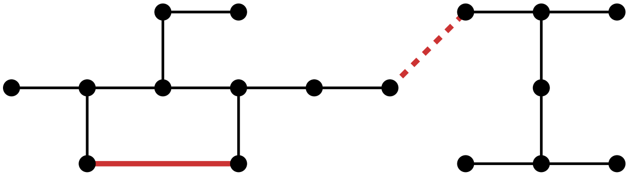

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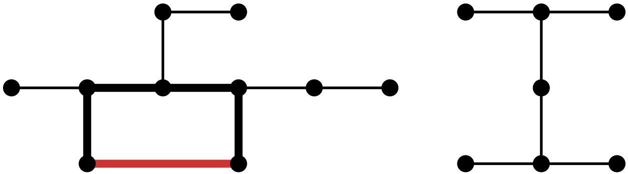

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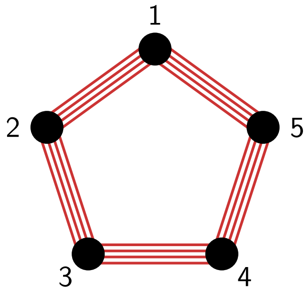

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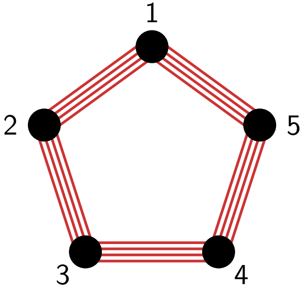

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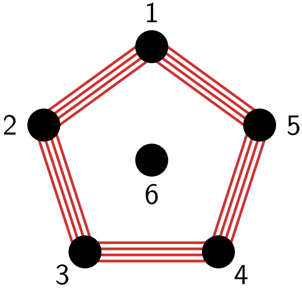

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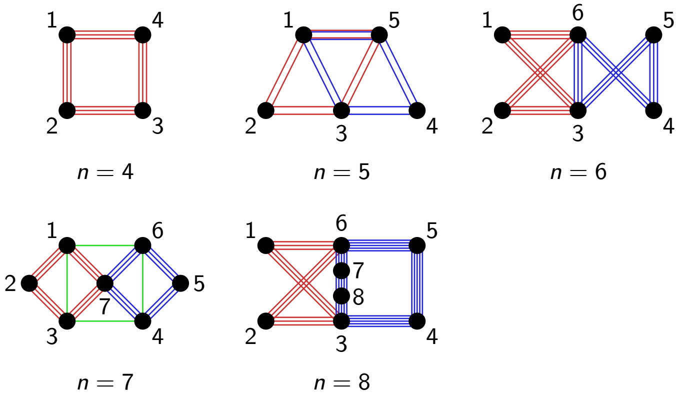

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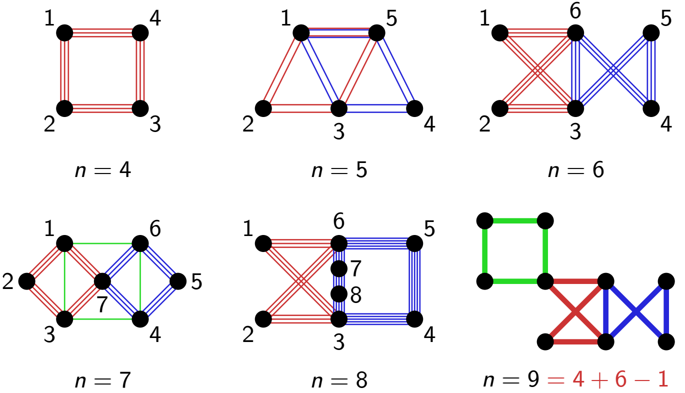

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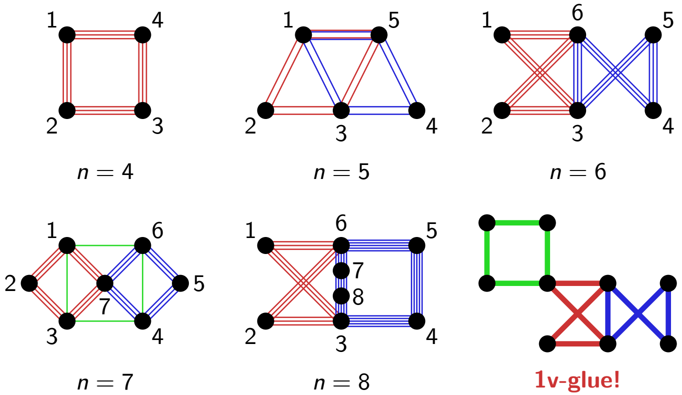

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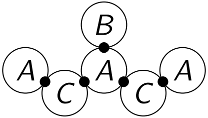

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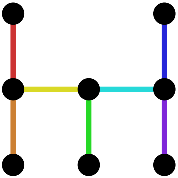
- A-graph: maximal rainbow tree



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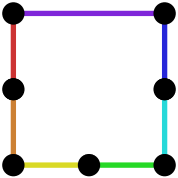

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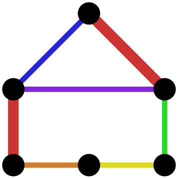
- A-graph: maximal rainbow tree
- B-graph: rainbow odd cycle, length ≥ 7
- C-graph: another subtle class



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