# **Saturation** around the *Happy Ending*

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May 30, 2024

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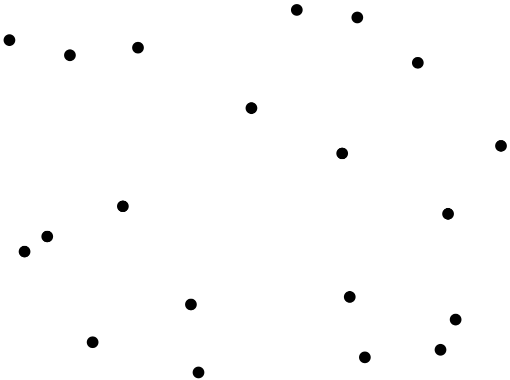

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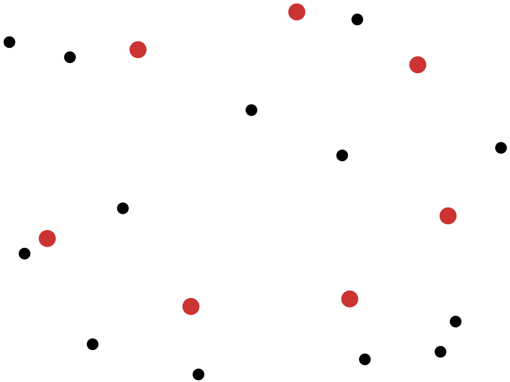

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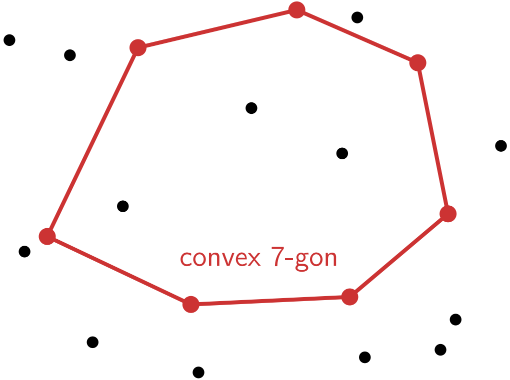

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- Klein's question again: $\text{ram}_{\text{g}}(n) < \infty$?

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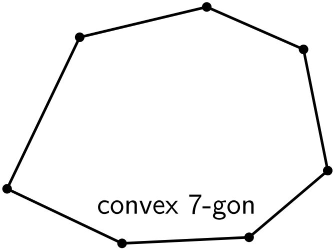

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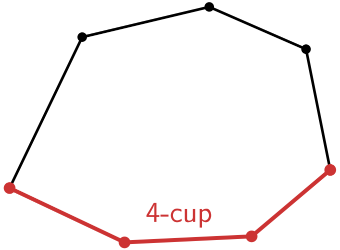

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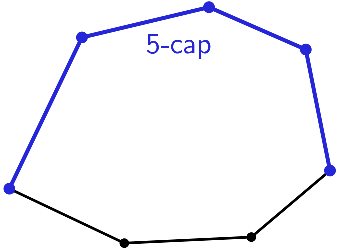

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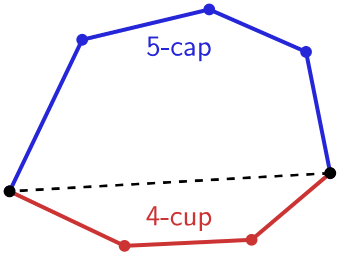

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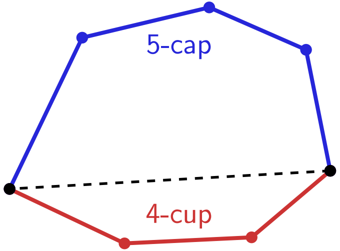


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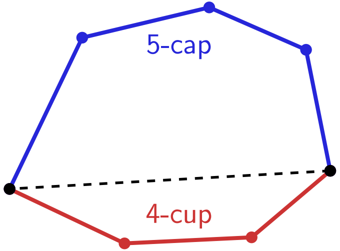


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**Theorem (Erdős–Szekeres, 1935)**

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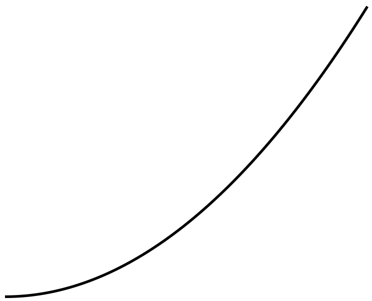

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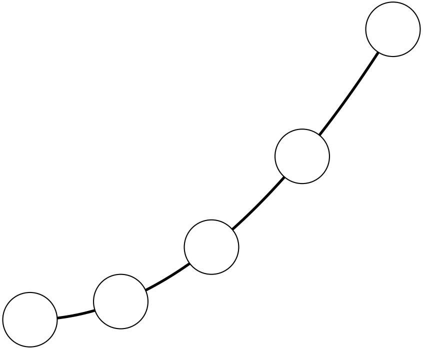

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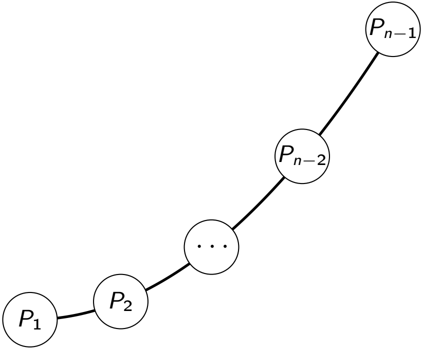


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- Question again: Is Erdős–Szekeres construction $n$-gon-saturated?

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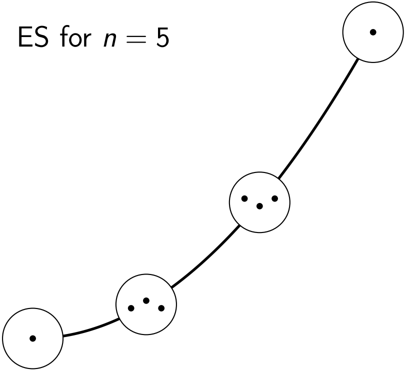
ES for $n = 5$



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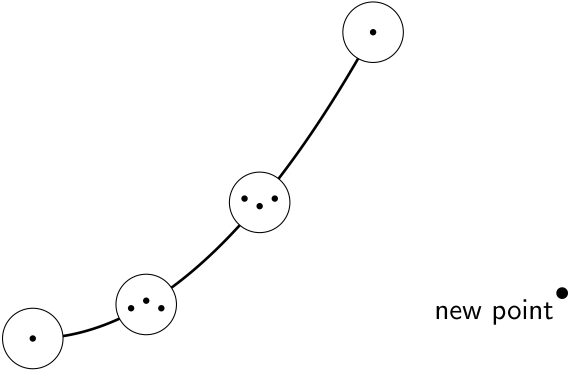

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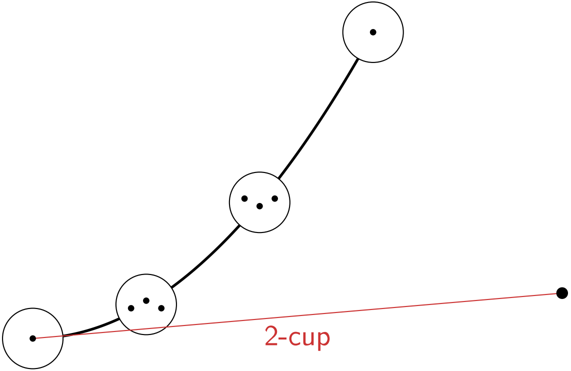

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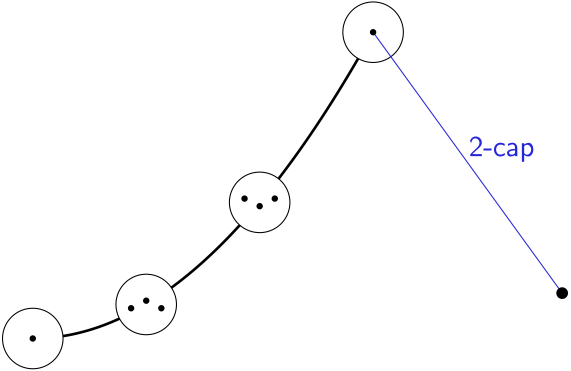

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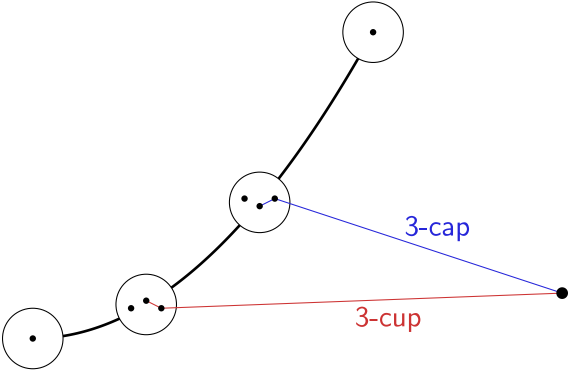

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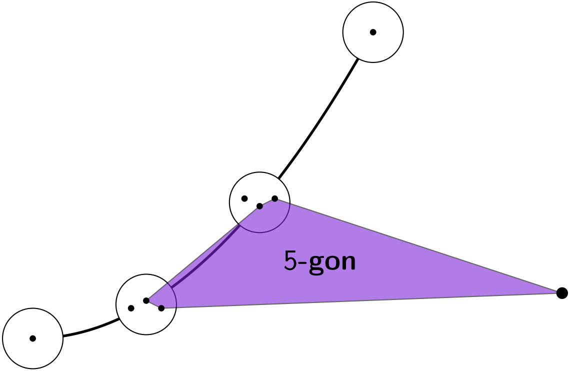

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- $(4, 5)$-cup-cap-saturated 8-point set:

$$(-60, 40) \quad (-40, 20) \quad (-20, 16) \quad (0, 10)$$

$$(5, -50) \quad (15, -40) \quad (25, -40) \quad (125, -230)$$

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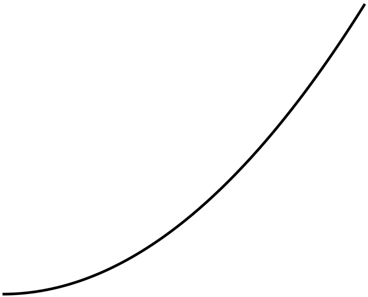

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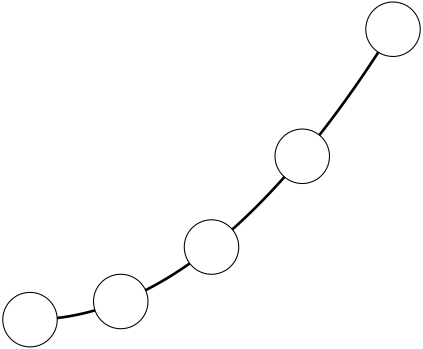

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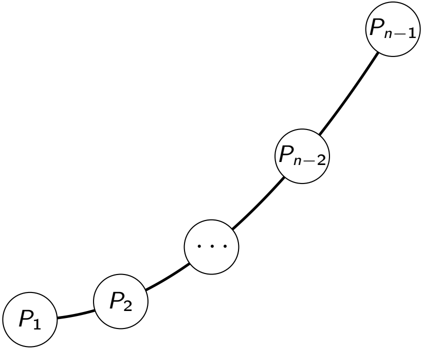


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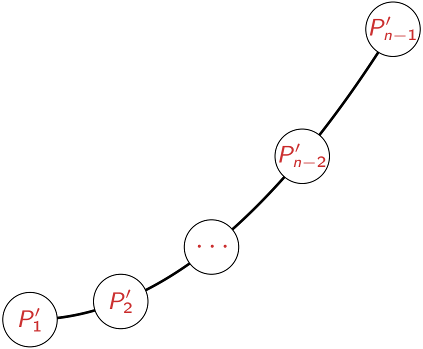


$P'_i$ is "flat" and $(i+1, n-i+1)$-cup-cap-sat.

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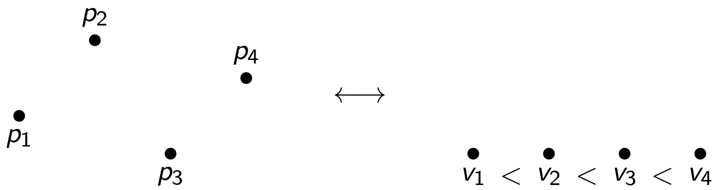

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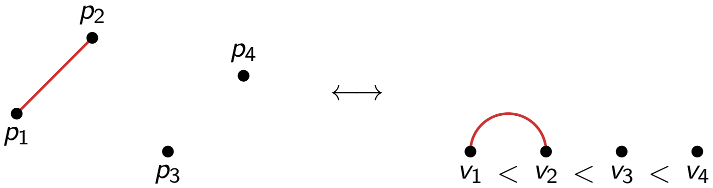

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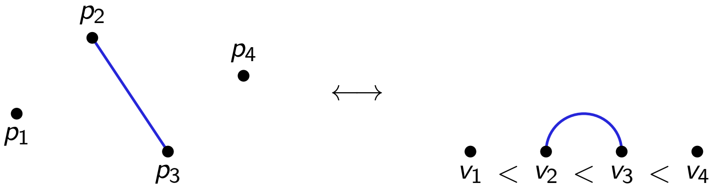

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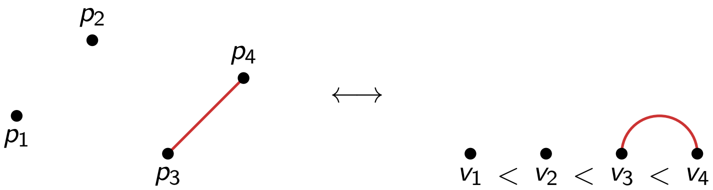

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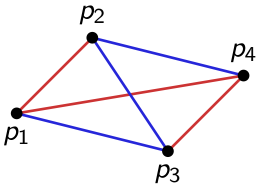


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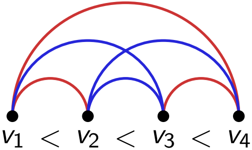

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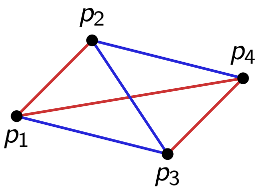


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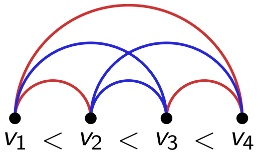


$k$-increasing subseq $\longleftrightarrow$ $(k-1)$-edge red path

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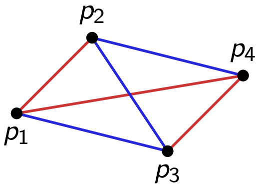
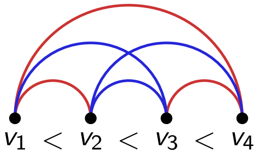



$\longleftrightarrow$

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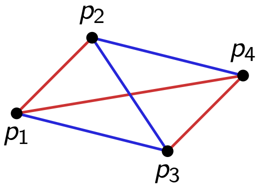


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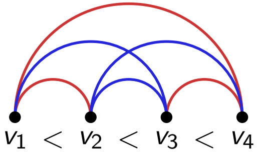


| | | |
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- In the right, any coloring induced from the left is "transitive"

# From geometry to graph: cup and cap

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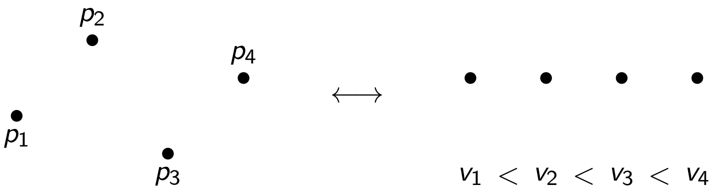

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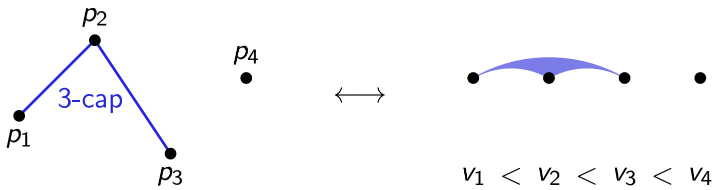

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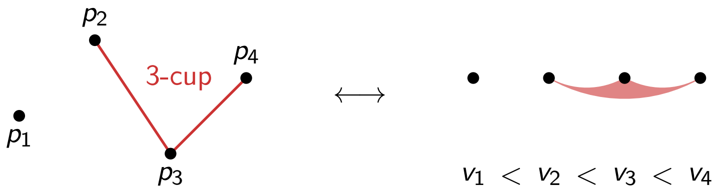

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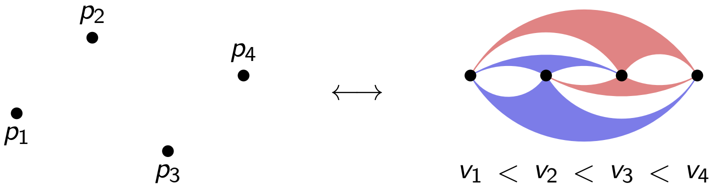

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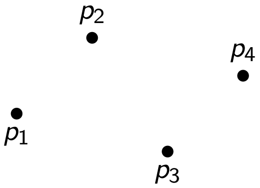


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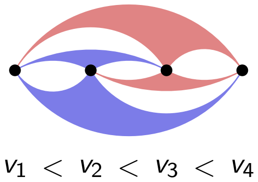


$k$-cup $\longleftrightarrow$ $(k-2)$-edge red path

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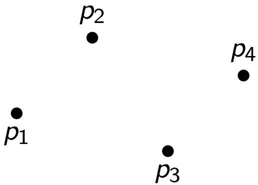


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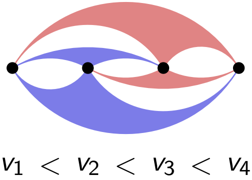


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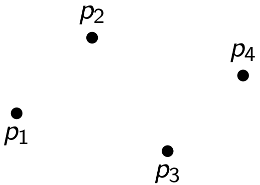


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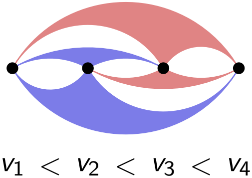


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- Monotone path = tight path on increasing vertices

# Monotone path problems

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