Saturation around the *Happy Ending*

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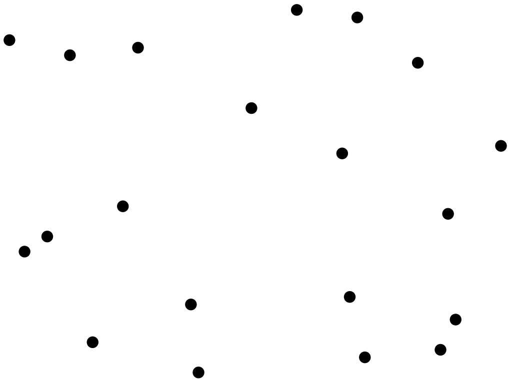

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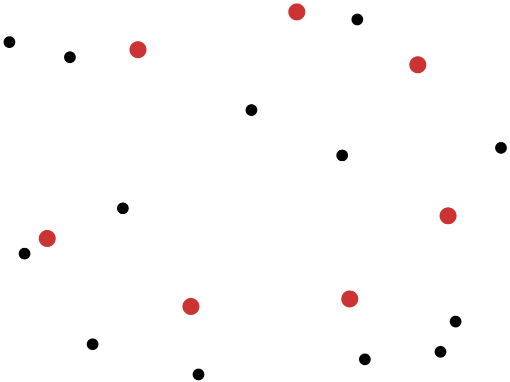

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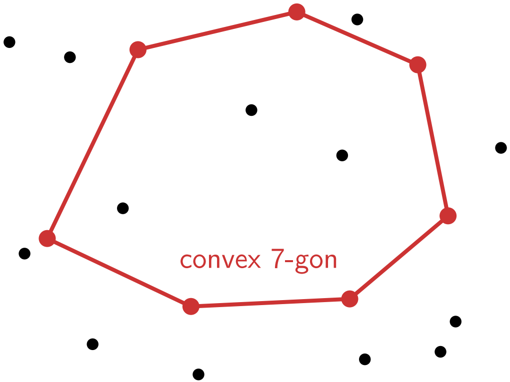

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In particular,

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- Klein's question again: $\mathrm{ram}_{\mathrm{g}}(n) < \infty$?

Erdős–Szekeres solution

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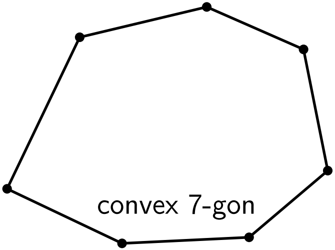

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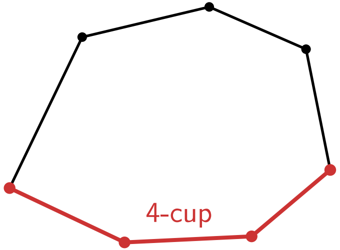

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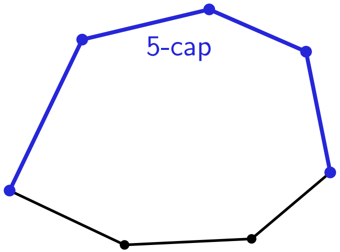

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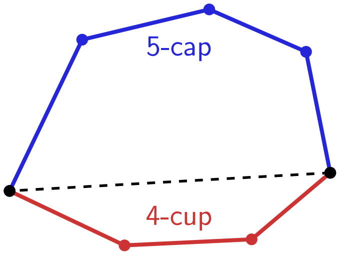

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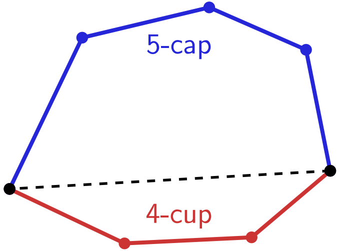

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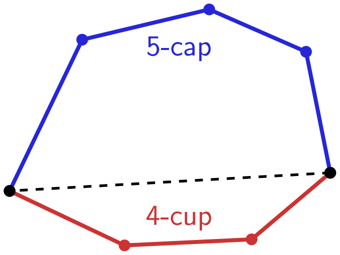

$\mathrm{ram_c}(k, \ell)$: number of points for k-cup or ℓ-cap

Theorem (Erdős–Szekeres, 1935)

$$\mathrm{ram_c}(k, \ell) = \binom{k+\ell-4}{k-2} \implies \mathrm{ram_g}(n) \le 4^{n-o(n)}$$

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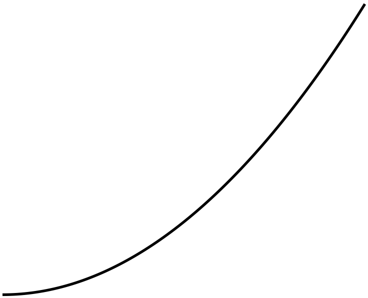

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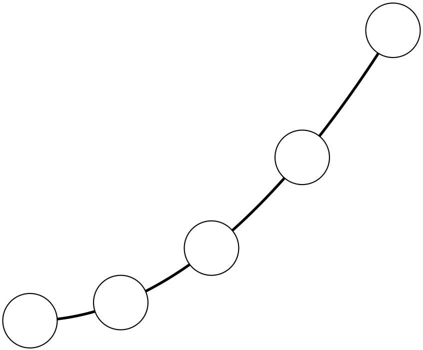

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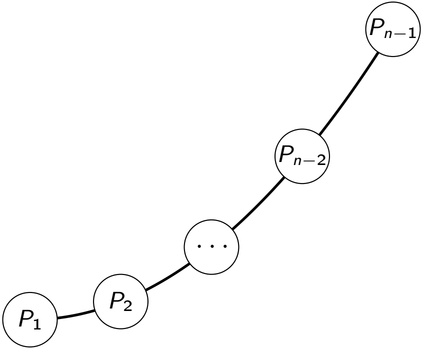

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P is n-gon-saturated: P avoids n-gon, but adding any point into P creates an n-gon

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- Question again: Is Erdős–Szekeres construction n-gon-saturated?

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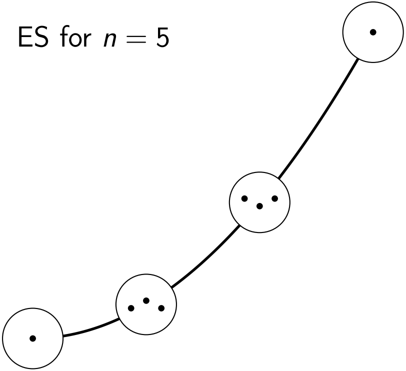

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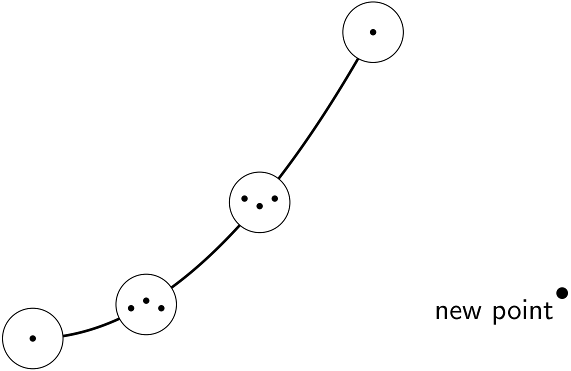

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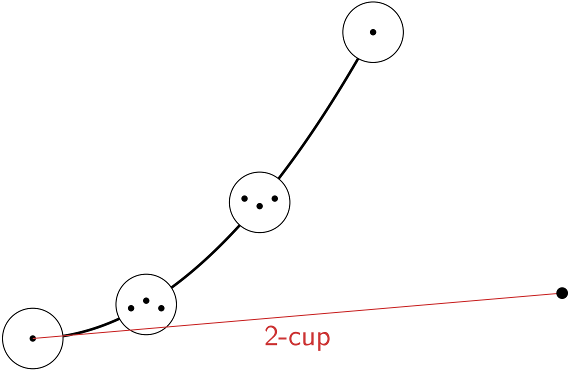

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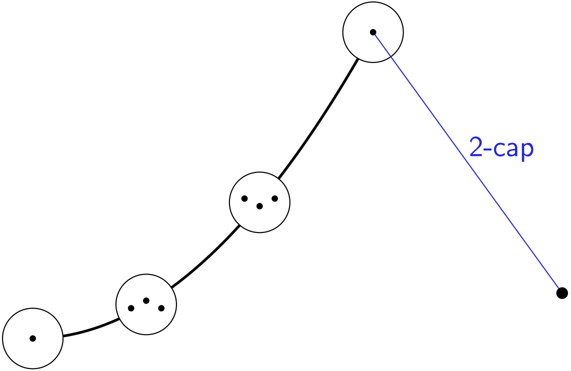

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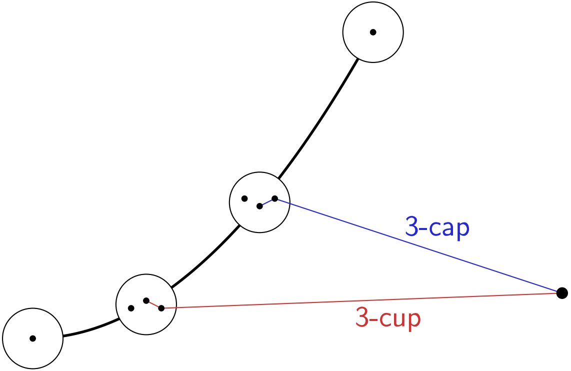

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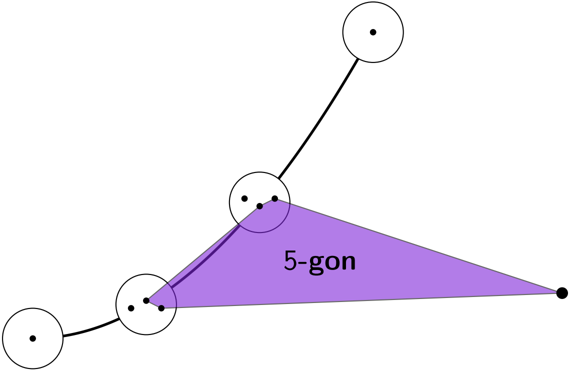

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- $(4, 5)$-cup-cap-saturated 8-point set:

$$(-60, 40) \quad (-40, 20) \quad (-20, 16) \quad (0, 10)$$

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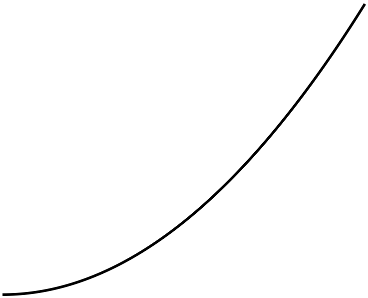

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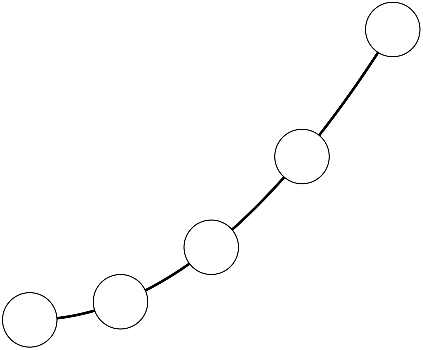

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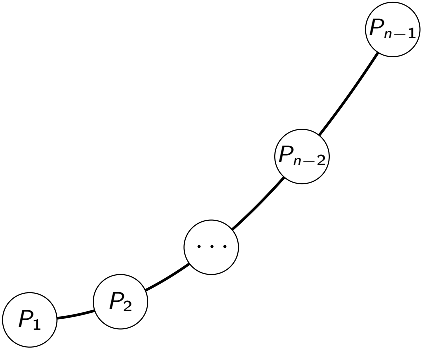

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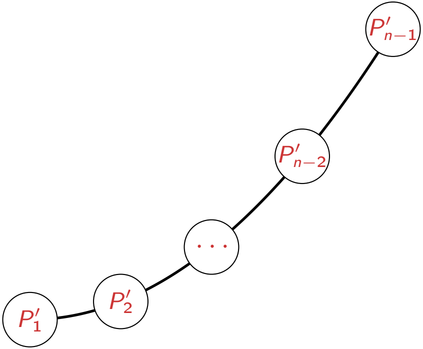

P'_i is "flat" and $(i+1, n-i+1)$-cup-cap-sat.

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